Cholesky factorizations

Lecture 11

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Beck, sections 5.3

Newton's method and positive-definite matrices

For Hybrid Newton: need to determine if $\nabla^2 f \succeq 0$ or not.

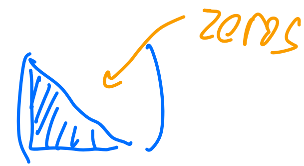
Cholesky factorizations are an efficient alternative to eigenvalue decompositions.

Cholesky factorization

__Theorem:__ Let $A \in \mathbb{R}^{n \times n}$ and symmetric. Then

$A > 0$ iff $\exists\, L \in \mathbb{R}^{n \times n}$ that is invertible and lower triangular s.t. $A = L L^T$.

L lower triangular: all entries above diagonal are 0

$L =$ 

$A = L L^T$ is a Cholesky factorization of A.

How are Cholesky decompositions computed? Gaussian elimination with one modification.

Ex $\underline{\underline{A}} = \begin{pmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{pmatrix}$ (from Wikipedia)

Perform Gaussian elimination, __but__ at each pivot, ~~multiply~~ divide pivot row by square root of pivot element.

$$\begin{pmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{pmatrix} \xrightarrow[r3 \leftarrow r3 + 4r1]{r2 \leftarrow r2 - 3r1} \begin{pmatrix} 4 & 12 & -16 \\ 0 & 1 & 5 \\ 0 & 5 & 34 \end{pmatrix}$$

$$\xrightarrow{r1 \leftarrow r1 \cdot \frac{1}{2}} \begin{pmatrix} 2 & 6 & -8 \\ 0 & 1 & 5 \\ 0 & 5 & 34 \end{pmatrix} \xrightarrow{r3 \leftarrow r3 - 5r2} \begin{pmatrix} 2 & 6 & -8 \\ 0 & 1 & 5 \\ 0 & 0 & 9 \end{pmatrix}$$

$$\xrightarrow{r2 \leftarrow \frac{1}{1} \cdot r2} \begin{pmatrix} 2 & 6 & -8 \\ 0 & 1 & 5 \\ 0 & 0 & 9 \end{pmatrix} \xrightarrow{r3 \leftarrow \frac{1}{3} \cdot r3} \begin{pmatrix} 2 & 6 & -8 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{pmatrix}$$

Then: $\underline{\underline{L}} = \begin{pmatrix} 2 & 6 & -8 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{pmatrix}$, and $\underline{\underline{A}} = \underline{\underline{L}}\,\underline{\underline{L}}^T$

Therefore: $\underline{\underline{A}} \succ \underline{\underline{0}}$.

pivot

How would this procedure fail? If a diagonal element is 0 or negative during the operation.

If the procedure fails, there is no $\underline{\underline{L}}$ s.t. $\underline{\underline{A}} = \underline{\underline{L}}\,\underline{\underline{L}}^T$, and hence $\underline{\underline{A}} \succ \underline{\underline{0}}$ is not true.

So: Cholesky factorizations (success or failure) are efficient ways to determine if $\underline{\underline{A}} \succ \underline{\underline{0}}$ is true.

<u>PSA</u>: Do <u>not</u> implement Cholesky factorizations yourself. Use canned routines.

Linear algebraic utility of Cholesky factorization

Cholesky factorizations in Newton's method

In hybrid Newton: ascertain $\nabla^2 f(\underline{x}_k) \succ \underline{0}$ by computing (or attempting to compute) a Cholesky factorization for $\nabla^2 f(\underline{x}_k)$.