Definite matrices

Lecture 05

September 9, 2021

Beck, section 2.2

Definite matrices

Definition: Let $\underline{\underline{A}} \in \mathbb{R}^{n\times n}$ be symmetric.

(i) $\underline{\underline{A}}$ is ("strictly") positive definite if $\underline{x}^T \underline{\underline{A}}\, \underline{x} > 0 \ \forall\, \underline{x} \in \mathbb{R}^n$ with $\underline{x} \neq \underline{0}$.

(ii) $\underline{\underline{A}}$ is positive semi-definite if $\underline{x}^T \underline{\underline{A}}\, \underline{x} \geq 0 \ \forall\, \underline{x} \in \mathbb{R}^n$

(iii) $\underline{\underline{A}}$ is ("strictly") negative definite if $\underline{x}^T \underline{\underline{A}}\, \underline{x} < 0 \ \forall\, \underline{x} \in \mathbb{R}^n$ with $\underline{x} \neq \underline{0}$

(iv) $\underline{\underline{A}}$ is negative semi-definite if $\underline{x}^T \underline{\underline{A}}\, \underline{x} \leq 0 \ \forall\, \underline{x} \in \mathbb{R}^n$.

(v) $\underline{\underline{A}}$ is indefinite if $\exists\ \underline{x}, \underline{y}$ s.t. $\underline{x}^T \underline{\underline{A}}\, \underline{x} < 0$ and

$\underline{y}^T \underline{\underline{A}}\, \underline{y} > 0$.

Examples

Example

Show that

$$\boldsymbol{A} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

is positive-definite.

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \underline{x}^T \underline{\underline{A}}\, \underline{x} = (x_1 \quad x_2) \begin{pmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 \end{pmatrix}$$

$$= 2x_1^2 - x_1 x_2 - x_1 x_2 + 2x_2^2$$

$$= x_1^2 - 2x_1 x_2 + x_2^2 + x_1^2 + x_2^2$$

$$= (x_1 - x_2)^2 + x_1^2 + x_2^2 \geq 0$$

It equals 0 iff $x_1 = x_2 = 0$.

$\Rightarrow \underline{x}^T \underline{A}\, \underline{x} > 0 \quad \forall\ \underline{x} \neq \underline{0}$.

Examples

Example

Show that

$$\boldsymbol{A} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

is positive-definite.

Example

Show that

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},$$

is not definite (i.e., "indefinite").

$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\quad \underline{x}^T \underline{\underline{A}}\, \underline{x} = (x_1 \quad x_2)\begin{pmatrix} x_1 + 2x_2 \\ 2x_1 + x_2 \end{pmatrix}$

$$= x_1^2 + 2x_1x_2 + 2x_1x_2 + x_2^2$$

$$= (x_1 + x_2)^2 + 2x_1x_2$$

Eg. $\underline{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \underline{x}^T \underline{\underline{A}}\, \underline{x} > 0$

$\underline{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \underline{x}^T \underline{\underline{A}}\, x < 0.$

What does definiteness give us?

If $\underline{\underline{A}}$ is positive-def, then $f_{\underline{\underline{A}}}(\underline{x}) : \mathbb{R}^n \to \mathbb{R}$

$$f_{\underline{\underline{A}}}(\underline{x}) = \underline{x}^T \underline{\underline{A}}\, \underline{x}.$$

The graph of $f_{\underline{\underline{A}}}$ is an upward-facing bowl.

Diagonal elements of in/definite matrices

Proposition: Assume $\underline{\underline{A}}$ is symmetric, $n \times n$.

(i) if $\underline{\underline{A}}$ is positive definite (semi-definite) then the diagonal elements of $\underline{\underline{A}}$ are all positive (non-negative).

(ii) if $\underline{\underline{A}}$ is negative definite (semi-definite) then the diagonal elements of $\underline{\underline{A}}$ are all negative (non-positive).

Idea of proof: $\underline{e}_i = (0, 0 \ldots, 1, 0, \ldots 0)$

ith entry

Note that $\underline{e}_i^T \underline{\underline{A}}\, \underline{e}_i = A_{i,i}$

Proposition: If $\underline{\underline{A}}$ is positive (semi-) definite, then $-\underline{\underline{A}}$ is negative (semi-) definite.

Notation: "$\underline{\underline{A}}$ is positive definite" is written $\underline{\underline{A}} \succ \underline{\underline{0}}$

"$\underline{\underline{A}}$ is positive semi-definite" is written $\underline{\underline{A}} \succeq \underline{\underline{0}}$

Similarly for $\prec$, $\preceq$

The spectrum of definite matrices

Thm: Let $\underline{\underline{A}}$ be symmetric $n \times n$.

(i) $\underline{\underline{A}}$ is positive definite iff all eigenvalues of $\underline{\underline{A}}$ are positive

(ii) $\underline{\underline{A}}$ is positive semi-definite iff all eigenvalues of $\underline{\underline{A}}$ are non-negative.

(iii)
(iv) } Similarly for negative (semi-)definite matrices.

Proof (of (i), definite)

$$\underline{\underline{A}} \succ \underline{\underline{0}} \quad \text{iff} \quad \underline{x}^T \underline{\underline{A}}\, \underline{x} > 0 \quad \forall\, \underline{x} \neq \underline{0}$$

$\underline{\underline{A}}$ symmetric $\Rightarrow$ $\underline{\underline{A}} = \underline{\underline{U}}\,\underline{\underline{\Lambda}}\,\underline{\underline{U}}^T$, $\underline{\underline{U}}$: orthogonal

$$\underline{\underline{\Lambda}} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \qquad \lambda_j : \text{eigenvalues of } \underline{\underline{A}}.$$

$$\underline{x}^T \underline{\underline{A}}\,\underline{x} = \underline{x}^T \underline{\underline{U}}\,\underline{\underline{\Lambda}}\,\underbrace{\underline{\underline{U}}^T \underline{x}}_{\text{"}y\text{"}} = \underline{y}^T \underline{\underline{\Lambda}}\,\underline{y}$$

$$\underline{x}^T \underline{\underline{A}}\,\underline{x} > 0 \quad \text{iff} \quad \sum_{j=1}^{n} \lambda_j y_j^2 > 0$$

if $\underline{x} \neq \underline{0} \Rightarrow \underline{\underline{U}}^T \underline{x}$ satisfies $\|\underline{\underline{U}}^T x\|_2^2 = \underline{x}^T \underline{\underline{U}}\,\underline{\underline{U}}^T \underline{x}$

$$= \underline{x}^T \underline{x} = \|\underline{x}\|_2^2 > 0.$$

$$\Rightarrow y \neq 0.$$

if all $\lambda_j > 0 \Rightarrow \sum_{j=1}^{n} \lambda_j y_j^2 > 0$.

if $\exists\, \lambda_j \leq 0 \Rightarrow$ choose $\underline{x} = \underline{v}_j$ eigenvector associated to λ_j

$$\Rightarrow \underline{x}^T \underline{\underline{A}}\,\underline{x} = \lambda_j \|v_j\|_2^2 \leq 0$$

$\Rightarrow \underline{x}^T \underline{\underline{A}}\,\underline{x} > 0 \;\; \forall\, \underline{x} \neq 0$ iff $\lambda_j(\underline{\underline{A}}) > 0 \;\; \forall\, j = 1, .., n.$

<u>Corollary</u>: Let $\underline{\underline{A}}$ be symmetric. Then $\underline{\underline{A}}$ is indefinite iff $\exists\, \lambda_j(\underline{\underline{A}}) > 0$, $\lambda_k(\underline{\underline{A}}) < 0$.

Examples

Example

Recall that

$$\boldsymbol{A} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

is positive-definite. Compute the spectrum of $\boldsymbol{A}$.

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \det\begin{pmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{pmatrix} = (\lambda-2)^2 - 1 = 0$$

$$\lambda = 2 \pm 1 = 3,\ 1 > 0.$$

Examples

Example

Recall that

$$\boldsymbol{A} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

is positive-definite. Compute the spectrum of $\boldsymbol{A}$.

Example

Recall that

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},$$

is indefinite. Compute the spectrum of $\boldsymbol{A}$.

$\det(\underline{A} - \underline{I}\lambda) = (\lambda - 1)^2 - 4 = 0 \Rightarrow \lambda = 1 \pm 2 = 3, -1$

$\Rightarrow \underline{A}$ indefinite.

Diagonal matrices

$$\underline{\underline{D}} = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix} \in \mathbb{R}^{n \times n}$$

$\underline{\underline{D}}$ is positive (semi-)definite iff $d_j > 0$ ($d_j \geq 0$) $\forall j = 1 \ldots n$.

Similar for negative definite.

$\underline{\underline{D}}$ is indefinite iff $\exists\, d_j > 0,\ d_k < 0$.

Matrix square roots

Def: Let $\underline{\underline{A}} \succ \underline{\underline{0}}$ (and hence symmetric). Then $\exists$ unique $\underline{\underline{B}} \succ \underline{\underline{0}}$ s.t. $\underline{\underline{A}} = \underline{\underline{B}}\,\underline{\underline{B}} = \underline{\underline{B}}^2$. $\underline{\underline{B}}$ is the "square root" or "positive square root" of $\underline{\underline{A}}$.

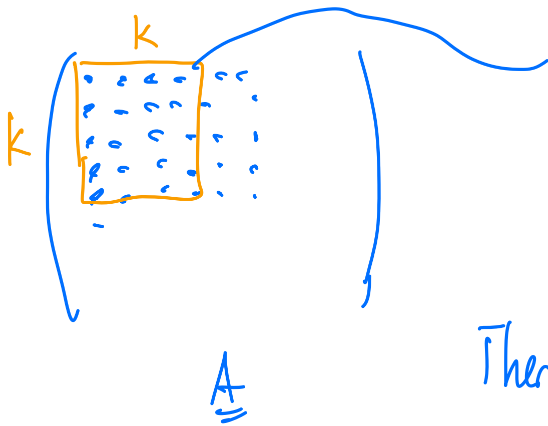
$\underline{\underline{B}} = ?$

$$\underline{\underline{A}} = \underline{\underline{U}}\,\underline{\underline{\Lambda}}\,\underline{\underline{U}}^T \qquad \underline{\underline{B}} = \underline{\underline{U}}\sqrt{\underline{\underline{\Lambda}}}\,\underline{\underline{U}}^T \qquad \sqrt{\underline{\underline{\Lambda}}} = \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix}$$

Can verify: $\underline{\underline{B}}\,\underline{\underline{B}} = \underline{\underline{A}}$ ✓, and $\underline{\underline{B}} \succ \underline{\underline{0}}$

Identifying definite matrices

Let $\underline{\underline{A}} \in \mathbb{R}^{n \times n}$ be symmetric. A "principal minor" of $\underline{\underline{A}}$ is a determinant of the upper left $k \times k$ block of $\underline{\underline{A}}$ $(k \leq n)$



$D_k(\underline{\underline{A}}) = \det(\text{upper-left } k \times k \text{ block of } \underline{\underline{A}})$.

There are n principal minors

<u>Lemma</u>: ~~If~~ If $\underline{\underline{A}}$ is symmetric, then $\underline{\underline{A}} \succ \underline{\underline{0}}$ iff $D_k(\underline{\underline{A}}) > 0 \ \forall$ $k = 1, \ldots n$.

General properties of matrices:

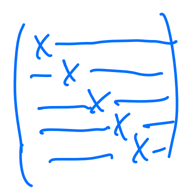
$$\det(\underline{\underline{A}}) = \prod_{i=1}^{n} \lambda_i$$

$$tr(\underline{\underline{A}}) = \sum_{i=1}^{n} \lambda_i$$

If $\underline{\underline{A}} \succ \underline{\underline{0}}$ then $\det(\underline{\underline{A}}) > 0$, $tr(\underline{\underline{A}}) > 0$.

<u>Lemma</u>: If $\underline{\underline{A}} \in \mathbb{R}^{2\times 2}$ and is symmetric: $\underline{\underline{A}} \succ \underline{\underline{0}}$ iff $\det(\underline{\underline{A}}) > 0$, $tr(\underline{\underline{A}}) > 0$.

(New day)



$\underline{\underline{A}}$

<u>Definition</u>: Let $\underline{\underline{A}} \in \mathbb{R}^{n\times n}$.

(i) $\underline{\underline{A}}$ is diagonally dominant if $|A_{i,i}| \geq \sum_{j\neq i} |A_{i,j}|$ $\forall\ i = 1 .. n$

(ii) $\underline{\underline{A}}$ is strictly diagonally dominant if

$$|A_{i,i}| > \sum_{j\neq i} |A_{i,j}| \quad \forall\ i = 1, .. n.$$

Thm: Let $\underline{\underline{A}} \in \mathbb{R}^{n \times n}$ be symmetric.

$A_{i,i} \geq 0$

(i) If $\cancel{|}A_{i,i}\cancel{|} \geq 0 \;\; \forall\, i = 1, \ldots, n$ and $\underline{\underline{A}}$ is diagonally dominant, then $\underline{\underline{A}} \succeq \underline{\underline{0}}$

(ii) If $A_{i,i} > 0 \;\; \forall\, i = 1, \ldots, n$, and $\underline{\underline{A}}$ is strictly diagonally dominant, then $\underline{\underline{A}} \succ \underline{\underline{0}}$.