HW #1 due Tuesday.

Basic Topology

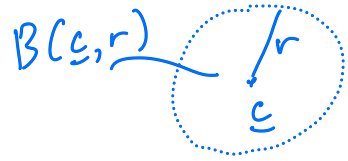
Lecture ~~02~~ 03

September 2, 2021

Beck, sections 1.5

Open and closed balls

In $\mathbb{R}^n$:



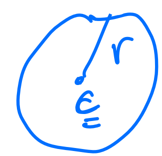
Given $\underline{c} \in \mathbb{R}^n$, $r > 0$,

$$B(\underline{c}, r) = \{ \underline{x} \in \mathbb{R}^n \mid \| \underline{x} - \underline{c} \| < r \}$$

"open" ball

any norm.

"closed" ball

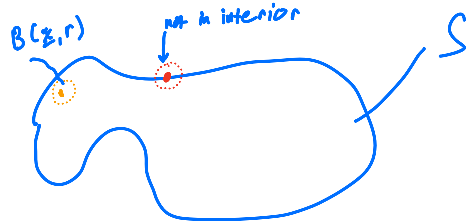


$$B[\underline{c}, r] = \{ \underline{x} \in \mathbb{R}^n \mid \| \underline{x} - \underline{c} \| \le r \}$$

The set interior and open sets

Given a set $S \subset \mathbb{R}^n$, then $\underline{x} \in S$ is in the <u>interior</u> of S if $\exists\, r > 0$ s.t. $B(\underline{x}, r) \subset S$.

"is a subset of"



The interior of a set S, $\mathrm{int}(S) = S^\circ$, is the collection of all points in the interior of S.

$$\mathrm{int}(S) = \{\underline{x} \in S \mid \exists\, r > 0 \text{ s.t. } B(\underline{x}, r) \subset S\}$$

A set S is open if $S = \mathrm{int}(S)$.

Ex: in $\mathbb{R}$: $(0,1)$ is open

$(-\pi, \infty)$ is open

$[3, \pi)$ is not open

$\text{int}([3,\pi)) = (3,\pi)$.

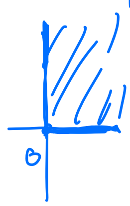
$\mathbb{R}$ is open

in $\mathbb{R}^n$: $\mathbb{R}^n$ is open

$B(\underline{x}, r)$ is open $(r>0)$

$\text{int}(\mathbb{R}^n_+) = \mathbb{R}^n_{++}$

$\mathbb{R}^2_+$



Set boundaries

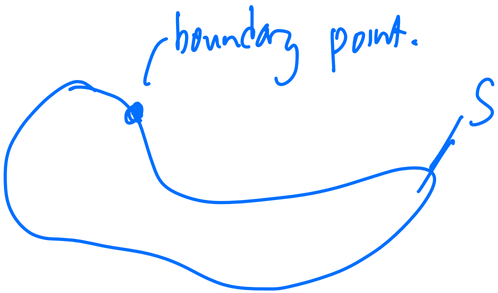
Given $S \subset \mathbb{R}^n$, $\underline{x} \in \mathbb{R}^n$ is on the boundary of S if

$\forall\, r > 0$, $\boxed{B(\underline{x}, r) \cap S \neq \emptyset \quad \text{and} \quad B(\underline{x}, r) \cap S^c \neq \emptyset}$

empty set

"complement" of S, all points not in S.

$B(\underline{x}, r)$ contains points in S and ~~not~~ not in S

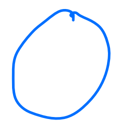


The set of all boundary points of S is denoted $bd(S) = \partial S$

Ex. in $\mathbb{R}$: $\operatorname{bd}((0,1)) = \{0, 1\}$

$\operatorname{bd}(\mathbb{R}) = \emptyset$

in $\mathbb{R}^2$: $B(\underline{0}, 1) \longrightarrow \operatorname{bd}(B(\underline{0}, 1)) = \{\underline{x} : \|x\| = 1\}$



$\mathbb{R}^2$

Closed sets

Given a set S, the closure of S is S along with its boundary.

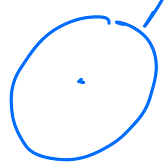
unioned with

$$cl(S) = S \cup bd(S)$$

$$cl(S) = \overline{S}$$

A set S is closed if $S = cl(S)$

$bd(B[\underline{c}, r])$



Ex: $B[\underline{c}, r]$ is closed

Proposition: If S is closed, then S^c is open.

E.g., $\bigcap_{n\geq 1}^{\infty} (0, 1+\frac{1}{n}) = [0, 1]$

A finite union/intersection of open/closed sets is open/closed.

<u>Ex</u>. $bd(\mathbb{R}) = \emptyset$.

$cl(\mathbb{R}) = \mathbb{R}$

Level sets and contours

Let $\underbrace{f : S \to \mathbb{R}}$, where $S \subset \mathbb{R}^n$

f is a function from S to $\mathbb{R}$.

Given $a \in \mathbb{R}$: $\mathrm{Lev}(f; a) = \{ \underline{x} \in S \mid f(\underline{x}) \leq a \}$

level set

$f^{-1}((-\infty, a])$

$\mathrm{Con}(f; a) = \{ \underline{x} \in S \mid f(\underline{x}) = a \}$

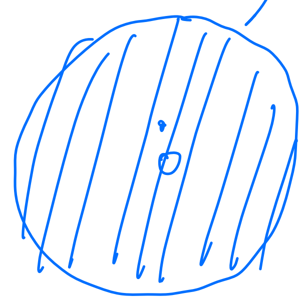
contour

$f^{-1}(a)$

Ex. $f: \mathbb{R}^2 \to \mathbb{R}$

$f(\underline{x}) = x_1^2 + x_2^2$

$\underline{x} = (x_1, x_2)$



$B[\underline{0}, \sqrt{a}]$
$\shortparallel$
$Lev(f; \sqrt{a})$

$Con(f; a)$: circle of radius $\sqrt{a}$ centered at origin

$bd(B(\underline{0}, \sqrt{a}))$

Proposition: If f is continuous, then $Lev(f; a)$ and $Con(f; a)$ are closed $\forall\, a \in \mathbb{R}$.

Bounded and compact sets

A set S is bounded if $\exists\, M \in \mathbb{R}_{++}$ s.t.

$$B(\underline{0}, M) \supset S$$

A set S is compact if it is bounded and closed.

(E.g. $B[\underline{c}, r]$ is compact).

Directional derivatives and the gradient

Let $f: S \to \mathbb{R}$, $S \subset \mathbb{R}^n$.

Assume that S is open.

Given $\underline{d} \neq \underline{0}$, the directional derivative of f at $\underline{x}$ in direction $\underline{d}$ is equal to

$$\lim_{t \downarrow 0} \frac{f(\underline{x} + t\underline{d}) - f(\underline{x})}{t},$$

if this limit exists.

We write $f'(\underline{x}; \underline{d})$ as this directional derivative.

If $\underline{d} = \underline{e}_j$, then we write $f'(\underline{x}; e_j) = \frac{\partial f}{\partial x_j}$

cardinal unit vector in jth direction

$$\underline{e}_j = (0, 0 \ldots 0, \underset{\substack{\uparrow \\ \text{entry } j}}{1}, 0 \ldots, 0) \in \mathbb{R}^n.$$

$\frac{\partial f}{\partial x_j}$ is a "partial" derivative.

$$\nabla f(\underline{x}) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(\underline{x}) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\underline{x}) \end{pmatrix} \in \mathbb{R}^n \qquad \nabla f: \text{"gradient"}$$

If ∇f exists at $\underline{x}$, we say f is differentiable at $\underline{x}$.

If f is differentiable at $\underline{x}$, then $f'(\underline{x}; \underline{d})$ exists $\forall\ \underline{d} \neq 0$, and

$$f'(\underline{x}; \underline{d}) = \nabla f^T(\underline{x})\,\underline{d} = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(\underline{x})\, d_j$$

Continuous differentiability

If $f: S \to \mathbb{R}^n$ is differentiable everywhere in S and ∇f is continuous on S, then f is continuously differentiable on S, $f \in C^1(S)$

$f \in C^k(S)$ if its kth derivatives are all continuous.

Consistency of directional derivatives

Proposition (First-order Taylor theorem)

Assume $f \in C^1(S)$, and let $\underline{x} \in S$. Then

$$\lim_{\|\underline{d}\| \downarrow 0} \frac{f(\underline{x}+\underline{d}) - f(x) - f'(\underline{x};\underline{d})}{\|\underline{d}\|} = 0$$

I.e.: $$\lim_{\|\underline{d}\| \downarrow 0} \frac{f(\underline{x}+\underline{d}) - f(\underline{x})}{\|\underline{d}\|} - \nabla f^T \frac{\underline{d}}{\|\underline{d}\|} = 0$$

I.e., for $\underline{y}$ "sufficiently" close to $\underline{x}$, then

$(\underline{y} = \underline{x} + \underline{d})$

$$f(\underline{y}) = f(\underline{x}) + \nabla f(\underline{x})^T (\underline{y} - \underline{x}) + o(\|\underline{y} - \underline{x}\|)$$

$o(\cdot)$ is a function satisfying

$$\frac{o(q)}{q} \to 0 \text{ as } q \downarrow 0.$$

The Hessian

The Hessian of $\underline{f}$ at $\underline{x}$ is defined as

$$H_f(\underline{x}) = \nabla^2 f(\underline{x}) = \underbrace{\begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & & & \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix}}_{\text{in } \mathbb{R}^{n \times n}}$$

First- and second-order Taylor Theorems

Theorem: If $f \in C^2(S)$ and $\underline{x} \in S$, $\underline{y} \in S$, then

$$f(\underline{y}) = f(\underline{x}) + \nabla f^T(\underline{x})(\underline{y}-\underline{x}) + \frac{1}{2}(\underline{y}-\underline{x})^T \nabla^2 f(\underline{x})(\underline{y}-\underline{x}) + o(\|\underline{y}-\underline{x}\|^2)$$