Math 5770/6640, MEEN 6025

HW 1 Solutions

1.1 $\underline{x} \in \mathbb{R}^n$, $\|\underline{x}\|_{1/2} = \left(\sum_{j=1}^{n} |x_j|^{1/2} \right)^2$

This function is positive and homogeneous but doesn't satisfy the triangle inequality:

$$\|\underline{x}+\underline{y}\|_{1/2} \not\leq \|\underline{x}\|_{1/2} + \|\underline{y}\|_{1/2}$$

E.g., $n=2$, $\underline{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\underline{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\left.\begin{array}{l} \|\underline{x}+\underline{y}\|_{1/2} = (1+1)^2 = 4 \\ \|\underline{x}\|_{1/2} = \|\underline{y}\|_{1/2} = 1 \end{array}\right\} \|\underline{x}+\underline{y}\|_{1/2} = 4 \not\leq 2 = \|\underline{x}\|_{1/2} + \|\underline{y}\|_{1/2}$$

1.3 $\underline{x}, \underline{y}, \underline{z} \in \mathbb{R}^n$, $\|\cdot\|$ is a norm

$$\|\underline{x}-\underline{z}\| = \|(\underline{x}-\underline{y}) + (\underline{y}-\underline{z})\| \leq \|\underline{x}-\underline{y}\| + \|\underline{y}-\underline{z}\|$$

triangle inequality

1.5 $\mathbb{R}^m$, norm $\|\cdot\|_b$

$\mathbb{R}^n$, norm $\|\cdot\|_a$

$\underline{\underline{A}} \in \mathbb{R}^{m\times n}$, $\|\underline{\underline{A}}\|_{a,b} = \max_{\|\underline{x}\|_a \leq 1} \|\underline{\underline{A}}\,\underline{x}\|_b$

Both $\|\cdot\|_a$ and $\|\cdot\|_b$ satisfy the triangle inequality.

$\underline{\underline{A}}, \underline{\underline{B}} \in \mathbb{R}^{m\times n} \implies \|(\underline{\underline{A}}+\underline{\underline{B}})\underline{x}\|_b = \|\underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{x}\|_b \leq \|\underline{\underline{A}}\,\underline{x}\|_b + \|\underline{\underline{B}}\,\underline{x}\|_b$

Therefore:

$$\|\underline{\underline{A}}+\underline{\underline{B}}\|_{a,b} = \max_{\|\underline{x}\|_a \le 1} \|(\underline{\underline{A}}+\underline{\underline{B}})\underline{x}\|_b \le \max_{\|\underline{x}\|_a \le 1} \left(\|\underline{\underline{A}}\,\underline{x}\|_b + \|\underline{\underline{B}}\,\underline{x}\|_b\right)$$

$$\le \max_{\|\underline{x}\|_a \le 1} \|\underline{\underline{A}}\,\underline{x}\|_b + \max_{\|\underline{x}\|_a \le 1} \|\underline{\underline{B}}\,x\|_b$$

$$= \|\underline{\underline{A}}\|_{a,b} + \|\underline{\underline{B}}\|_{a,b}.$$

1.8 $\underline{\underline{A}} \in \mathbb{R}^{m\times n}$

$$\|\underline{\underline{A}}\|_{a,b} = \max_{\|\underline{x}\|_a \le 1.} \|\underline{\underline{A}}\,\underline{x}\|_b.$$

(1) Since $\{\underline{x} \in \mathbb{R}^n : \|\underline{x}\|_a = 1\} \subset \{\underline{x} \in \mathbb{R}^n : \|\underline{x}\|_a \le 1\}$,

then $\displaystyle\max_{\|\underline{x}\|_a \le 1} \|\underline{\underline{A}}\,\underline{x}\|_b \ge \max_{\|\underline{x}\|_a = 1} \|\underline{\underline{A}}\,\underline{x}\|_b.$ $(\star)$

(2) Suppose $\underline{x}_*$ is a(ny) maximizer of $\|\underline{\underline{A}}\,\underline{x}\|_b$ over $\|\underline{x}\|_a \le 1$.

Then $\|\underline{x}_*\|_a = 1$.

<u>Proof</u>: suppose $\|\underline{x}_*\|_a < 1$ and $\underline{x}_*$ maximizes $\|\underline{\underline{A}}\,\underline{x}\|_b$ $\forall \|\underline{x}\|_a \le 1$.

Then $y = \underline{x}_* / \|\underline{x}\|_*$ satisfies $\|y\|_a = 1$, and

$\|\underline{\underline{A}}\,y\|_b = \dfrac{1}{\|\underline{x}_*\|_a} \|\underline{\underline{A}}\,\underline{x}_*\|_b > \|\underline{\underline{A}}\,\underline{x}_*\|_b$, contradicting

the assumption that $\underline{x}_*$ maximizes $\|\underline{\underline{A}}\,\underline{x}\|_b$.

Thus, $\|\underline{x}_*\|_a = 1$.

(3) Let $\underline{x}_*$ be any maximizer of $\|\underline{A}\,\underline{x}\|_b$ over $\|\underline{x}\|_a \le 1$.
We know that $\|\underline{x}_*\|_a = 1$.

Then:

$$\max_{\|\underline{x}\|_a \le 1} \|\underline{A}\,\underline{x}\|_b = \|\underline{A}\,\underline{x}_*\|_b \le \max_{\|\underline{x}\|_a = 1} \|\underline{A}\,\underline{x}\|_b \qquad (\star\star)$$

$\underline{x}_*$ is just one vector satisfying $\|\underline{x}\|_a = 1$.

Combining $(\star)$ and $(\star\star)$ shows that

$$\max_{\|\underline{x}\|_a \le 1} \|\underline{A}\,\underline{x}\|_b = \max_{\|\underline{x}\|_a = 1} \|\underline{A}\,\underline{x}\|_b.$$

<u>1.9</u> $\underline{A} \in \mathbb{R}^{m \times n}$, $\|\cdot\|_a$ a norm on $\mathbb{R}^n$
$\|\cdot\|_b$ a norm on $\mathbb{R}^m$

$$\max_{\underline{x} \ne \underline{0}} \frac{\|\underline{A}\,\underline{x}\|_b}{\|\underline{x}\|_a} = \max_{\underline{x} \ne \underline{0}} \left\| \underline{A} \left(\frac{\underline{x}}{\|\underline{x}\|_a} \right) \right\|_b \le \max_{\|\underline{y}\|_a = 1} \|\underline{A}\,\underline{y}\|_b$$

$\frac{\underline{x}}{\|\underline{x}\|_a}$ always has norm 1.

and:

$$\max_{\underline{x} \ne \underline{0}} \frac{\|\underline{A}\,\underline{x}\|_b}{\|\underline{x}\|_a} \ge \max_{\|\underline{x}\|_a = 1} \frac{\|\underline{A}\,\underline{x}\|_b}{\|\underline{x}\|_a} = \max_{\|\underline{x}\|_a = 1} \|\underline{A}\,\underline{x}\|_b$$

$\{\underline{x} : \underline{x} \ne \underline{0}\} \supset \{\underline{x} : \|\underline{x}\|_a = 1\}$.

Thus. $$\max_{\underline{x}\neq \underline{0}} \frac{\|\underline{\underline{A}}\,\underline{x}\|_b}{\|\underline{x}\|_a} = \max_{\|\underline{x}\|_a=1} \|\underline{\underline{A}}\,\underline{x}\|_b \underset{\text{problem 1.8.}}{=} \|\underline{\underline{A}}\|_{a,b}$$

<u>1.13</u> $\underline{\underline{A}} \in \mathbb{R}^{m\times n}$

(i) $\|A\| = \lambda_{max}(\underline{\underline{A}}^T\underline{\underline{A}})$

$\|A^T\| = \lambda_{max}(\underline{\underline{A}}\,\underline{\underline{A}}^T)$

If $\|\underline{\underline{A}}\| = 0$, then $\underline{\underline{A}} = \underline{\underline{0}}$ (since $\|\cdot\|$ is a norm), so $\underline{\underline{A}}^T = \underline{\underline{0}}$ and $\|\underline{\underline{A}}^T\| = 0$.

So $\|\underline{\underline{A}}\| = \|\underline{\underline{A}}^T\|$ if $\underline{\underline{A}} = \underline{\underline{0}}$.

If $\|\underline{\underline{A}}\| \neq 0$, then $\lambda_{max}(\underline{\underline{A}}^T\underline{\underline{A}}) \neq 0$.

Claim: if λ is a nonzero eigenvalue of $\underline{\underline{A}}^T\underline{\underline{A}}$, then it is also an eigenvalue of $\underline{\underline{A}}\,\underline{\underline{A}}^T$.

Proof: $\exists\ \underline{v} \neq \underline{0}$ s.t. $$\underline{\underline{A}}^T\underline{\underline{A}}\,\underline{v} = \lambda \underline{v}, \quad \lambda \neq 0$$

$$(\underline{\underline{A}}\,\underline{\underline{A}}^T)(\underline{\underline{A}}\,\underline{v}) = \lambda(\underline{\underline{A}}\,\underline{v})$$

Note: $\underline{\underline{A}}\,\underline{v} \neq \underline{0}$ since if it did, then $\lambda = 0$.

So $\underline{y} = \underline{\underline{A}}\,\underline{v} \neq \underline{0}$, and

$$(\underline{\underline{A}}\,\underline{\underline{A}}^T)\underline{y} = \lambda \underline{y} \Rightarrow \lambda \text{ is an eigenvalue of } \underline{\underline{A}}\,\underline{\underline{A}}^T.$$

Therefore, $\underline{\underline{A}}\,\underline{\underline{A}}^T$ and $\underline{\underline{A}}^T\underline{\underline{A}}$ have the same set of nonzero eigenvalues, implying that if $\|\underline{\underline{A}}\| \neq 0$, then

$$\|\underline{\underline{A}}\| = \lambda_{max}(\underline{\underline{A}}^T\underline{\underline{A}}) = \lambda_{max}(\underline{\underline{A}}\,\underline{\underline{A}}^T) = \|\underline{\underline{A}}^T\|.$$

(ii) $\sum_{i=1}^{n} \lambda_i(\underline{\underline{A}}^T\underline{\underline{A}}) = tr(\underline{\underline{A}}^T\underline{\underline{A}}) = \sum_{i=1}^{n} (\underline{\underline{A}}^T\underline{\underline{A}})_{i,i}$

$\underline{\underline{A}} = \begin{pmatrix} | & | & & | \\ \underline{a}_1 & \underline{a}_2 & \cdots & \underline{a}_n \\ | & | & & | \end{pmatrix}$

$$= \sum_{i=1}^{n} \|\underline{a}_i\|_2^2$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} (A_{i,j})^2 = \|\underline{\underline{A}}\|_F^2.$$

1.14 $\underline{\underline{A}} \in \mathbb{R}^{n \times n}$, symmetric. Prove $\lambda_{max}(\underline{\underline{A}}) = \max_{\|\underline{x}\|=1} \underline{x}^T\underline{\underline{A}}\underline{x}$

One option: use results from Rayleigh quotient.

Second, direct strategy: $\underline{\underline{A}} = \underline{\underline{U}}\,\underline{\underline{\Lambda}}\,\underline{\underline{U}}^T$, $\underline{\underline{U}}$: orthogonal, $n \times n$

$\underline{\underline{\Lambda}}$: diagonal with entries $\lambda_1, \lambda_2, \ldots \lambda_n$

Let $\underline{x}$ satisfy $\|\underline{x}\| = 1$. Then $\underline{y} = \underline{\underline{U}}^T\underline{x}$ satisfies

$$\|\underline{y}\| = \sqrt{\underline{x}^T\underline{\underline{U}}\,\underline{\underline{U}}^T\underline{x}} = \sqrt{\underline{x}^T\underline{x}} = 1.$$

$$\underline{x}^T\underline{\underline{A}}\underline{x} = \underline{x}^T\underline{\underline{U}}\,\underline{\underline{\Lambda}}\,\underline{\underline{U}}^T\underline{x} = \underline{y}^T\underline{\underline{\Lambda}}\,\underline{y}$$

Since $\{\underline{x} \in \mathbb{R}^n \mid \|\underline{x}\| = 1\} = \{\underline{\underline{U}}^T\underline{x} \mid \|\underline{x}\| = 1\}$, then

$$\max_{\|\underline{x}\|=1} \underline{x}^T\underline{\underline{A}}\underline{x} = \max_{\|\underline{y}\|=1} \underline{y}^T\underline{\underline{\Lambda}}\,\underline{y} = \max_{\|\underline{y}\|=1} \sum_{j=1}^{n} \lambda_j y_j^2$$

$$= \max_{i=1,\ldots,n} \lambda_i = \lambda_{max}(\underline{\underline{A}}).$$

1.17: Define $A_n = (-\frac{1}{n}, \frac{1}{n})$, which is open $\forall\, n > 0$

Then $\bigcap_{n=1}^{\infty} A_n = \{0\}$, which is a closed set.

1.18 $A, B \subset \mathbb{R}^n$

Since $A \subset cl(A)$ and $B \subset cl(B)$, then

$A \cap B \subset cl(A) \cap cl(B)$

Intersections of closed sets are closed, so $cl(\bar{A}) \cap cl(\bar{B})$ is a closed set containing $A \cap B$.

$cl(A \cap B)$ is by definition the smallest closed set containing $A \cap B$, and thus is at least as small as $cl(\bar{A}) \cap cl(\bar{B})$, i.e.

$$cl(A \cap B) \subseteq cl(A) \cap cl(B)$$

Example in $\mathbb{R}$: $A = [0, 1)$, $B = (1, 2)$

$cl(A \cap B) = cl(\emptyset) = \emptyset$

$cl(A) \cap cl(B) = [0, 1] \cap [1, 2] = \{1\} \supset \emptyset$.