Polynomial approximation, II

MATH 6610 Lecture 26

November 13, 2020

Polynomial approximation

We've seen that polynomial interpolation is unisolvent. (in 1D)

Given a continuous f and distinct nodes $x_1, \ldots, x_{n+1}$, then the Lagrange form of the interpolant is

$$p(x) = \sum_{j=1}^{n+1} f(x_j) \ell_j(x), \qquad \ell_j(x) := \prod_{\substack{\ell=1,\ldots,n+1 \\ \ell \neq j}} \frac{x - x_\ell}{x_j - x_\ell}.$$

(Lagrange form)

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continuous functions on [a, b]

We'll need some notation:

$$P_n := \mathrm{span}\{1, x, \ldots, x^n\}, \quad C([a,b]) := \left\{ f : [a,b] \to \overset{\mathbb{R}}{\cancel{\infty}} \;\middle|\; f \text{is continuous on } [a,b] \right\},$$

and we metrize $C([a,b])$ with the norm,

$$\|f\| = \|f\|_\infty := \sup_{x \in [a,b]} |f(x)|.$$

Lebesgue functions and constants

To address accuracy, we investigate the *stability* of interpolation.

With $x_1, \ldots, x_{n+1}$ unique, fixed nodes, define $I_n : C([a,b]) \to P_n$ as the interpolation operator:

$$I_n f := \sum_{j=1}^{n+1} f(x_j)\ell_j(x).$$

(I_n is linear!)

(Note: I_n is a projector!) (I.e. $I_n I_n f = I_n f$)

Stability: $\|I_n\| = \sup_{f \neq 0} \frac{\|I_n f\|}{\|f\|} = \sup_{\|f\|=1} \|I_n f\|.$

induced norm

We can also look at stability of interpolation at a particular $x \in [a,b]$.

This notion of stability: $\sup_{\|f\|=1} |I_n f(x)|$

$$= \sup_{\|f\|=1} \left| \sum_{j=1}^{n+1} f(x_j)\, l_j(x) \right|$$

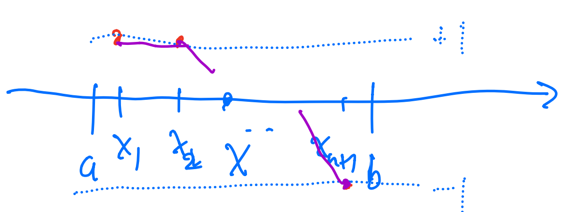
$$(\star) \le \sup_{\|f\|=1} \sum_{j=1}^{n+1} |f(x_j)| \cdot |l_j(x)|$$

$$(\star\star) \le \sum_{j=1}^{n+1} |l_j(x)|.$$

We can make this an equality:

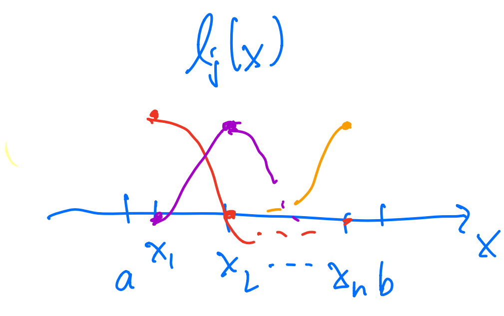
$(\star)$: choose $f(x_j) = \operatorname{sgn} l_j(x)$.

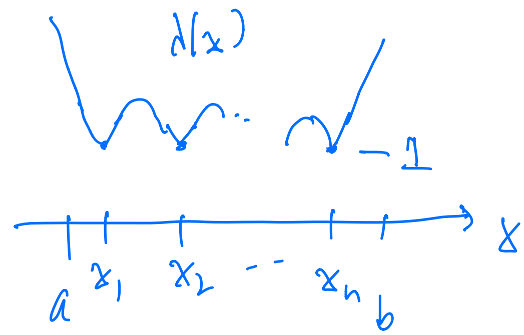
$(\star\star)$: pick f as pw-linear between x_j



So: $\sup_{\|f\|=1} |I_n f(x)| = \sum_{j=1}^{n+1} |\ell_j(x)| := \lambda(x)$

"Lebesgue function"





General interpolation stability:

$\|I_n\| = \sup_{\|f\|=1} \|I_n f\| = \sup_{\|f\|=1} \|\lambda(x)\|$

$= \|\lambda(x)\| := \Lambda$

"Lebesgue constant"

Lebesgue functions and constants

To address accuracy, we investigate the *stability* of interpolation.

With $x_1, \ldots, x_{n+1}$ unique, fixed nodes, define $I_n : C([a,b]) \to P_n$ as the interpolation operator:

$$I_n f := \sum_{j=1}^{n+1} f(x_j) \ell_j(x).$$

(Note: I_n is a projector!)

A computation shows that

$$\|I_n\|_{C([a,b]) \to P_n} = \Lambda := \|\lambda(x)\|_\infty ,$$

where

$$\lambda(x) := \sum_{j=1}^{n+1} |\ell_j(x)|$$

λ is called the Lebesgue function, and Λ the Lebesgue constant.

Stability gives us a way to investigate accuracy.

$$\|f - I_n f\| = \|f - p + p - I_n f\| \quad (p \in P_n)$$

$$\leq \|f-p\| + \|p - I_n f\|$$

$$= \|f-p\| + \|I_n p - I_n f\| \quad (p = I_n p)$$

$$= \|f-p\| + \|I_n (p-f)\| \quad (I_n \text{ is linear})$$

$$\leq \|f-p\| + \|I_n\| \cdot \|p-f\| \quad (\text{true for all induced norms})$$

$$= (1+\Lambda)\|f-p\|$$

$$\Longrightarrow \|f - I_n f\| \leq \underbrace{(1+\Lambda)}_{} \cdot \underbrace{\inf_{p \in P_n} \|f-p\|}_{}$$

depends only on nodes
independent of f.

best polynomial approx error.
independent of nodal choice.
(depends only on f, n).

Lebesgue's Lemma

With the operator norm of interpolation, here is a classical result quantifying the quality of polynomial interpolation:

Lemma (Lebesgue)

Let $f \in C([a,b])$, and assume that $x_1, \ldots, x_{n+1}$ are distinct nodes on $[a,b]$. Then

$$\|f - I_n f\|_\infty \leqslant \underbrace{(1 + \Lambda)}_{(1+\|I_n\|)} \inf_{p \in P_n} \|f - p\|_\infty$$

Thus, the Lebesgue constant governs the accuracy of polynomial interpolants.

Nodal sets

This raises the question: can we minimize Lebesgue constants?

Let Λ_n denote the Lebesgue constant for $n+1$ points.

Λ_n and Λ_{n+1} need not have common nodes.

(We have a general triangular array of nodes:

$$
\begin{array}{ccccc}
 & & x_{1,1} & & \leftarrow \Lambda_1 \\
 & x_{1,2} & & x_{2,2} & \leftarrow \Lambda_2 \\
x_{1,3} & & \cdots & x_{3,3} & \leftarrow \Lambda_3 \\
\vdots & & & \ddots & \vdots
\end{array}
$$

)

Nodal sets

This raises the question: can we minimize Lebesgue constants?

Let Λ_n denote the Lebesgue constant for $n+1$ points.

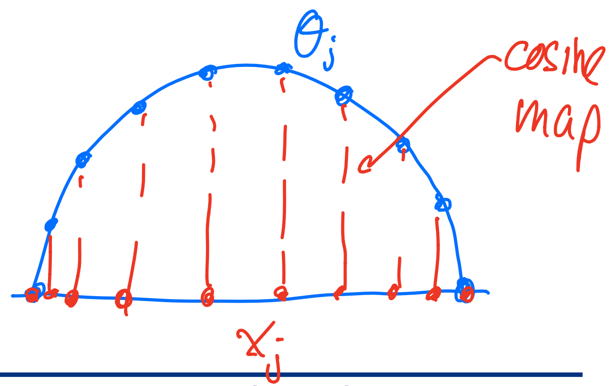
Λ_n and Λ_{n+1} need not have common nodes.

- For *any* triangular array of points, $\Lambda_n \overset{n\uparrow\infty}{\longrightarrow} \infty$.
- For equidistant nodes, $\Lambda_n \sim 2^n$. (This is bad!)
- For "Chebyshev" nodes , $\Lambda_n \sim \log n$. (This is asymptotically optimal)

Chebyshev nodes x_j (say on $[-1,1]$) are equidistant nodes under the cosine map:

$$x_j = \cos\theta_j, \qquad \theta_j = \frac{2j-1}{2n+2}\pi, \qquad j = 1,\ldots,n+1.$$

(clusters nodes at boundary)



Another error estimate

There are alternative strategies to computing error estimates for interpolation.

With $x_1, \ldots, x_{n+1}$ the nodes, the *nodal polynomial* $\omega(x)$ is

$$\omega(x) := \prod_{j=1}^{n+1} (x - x_j).$$

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There are alternative strategies to computing error estimates for interpolation.

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$$\omega(x) := \prod_{j=1}^{n+1} (x - x_j).$$

This can be used to construct an error estimate:

Theorem

Let $x_1, \ldots, x_{n+1}$ be distinct nodes on $[a, b]$. Then

$$|f(x) - [I_n f](x)| = \frac{\left| f^{(n+1)}(\xi) \right|}{(n+1)!} |\omega(x)|,$$

where $\xi = \xi(x)$ lies in $[a, b]$.