Polynomial approximation, II

MATH 6610 Lecture 26

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Polynomial approximation

We've seen that polynomial interpolation is unisolvent.

Given a continuous f and distinct nodes $x_1, \ldots, x_{n+1}$, then the Lagrange form of the interpolant is

$$p(x) = \sum_{j=1}^{n+1} f(x_j)\ell_j(x), \qquad \ell_j(x) := \prod_{\substack{\ell=1,\ldots,n+1 \\ \ell \neq j}} \frac{x - x_\ell}{x_j - x_\ell}.$$

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We'll need some notation:

$$P_n := \mathrm{span}\{1, x, \ldots, x^n\}, \quad C([a,b]) := \left\{ f : [a,b] \to \infty \;\middle|\; f \text{is continuous on } [a,b] \right\},$$

and we metrize $C([a,b])$ with the norm,

$$\|f\| = \|f\|_\infty := \sup_{x \in [a,b]} |f(x)|.$$

Lebesgue functions and constants

To address accuracy, we investigate the *stability* of interpolation.

With $x_1, \ldots, x_{n+1}$ unique, fixed nodes, define $I_n : C([a,b]) \to P_n$ as the interpolation operator:

$$I_n f := \sum_{j=1}^{n+1} f(x_j) \ell_j(x).$$

(Note: I_n is a projector!)

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A computation shows that

$$\|I_n\|_{C([a,b])\to P_n} = \Lambda := \|\lambda(x)\|_\infty,$$

where

$$\lambda(x) := \sum_{j=1}^{n+1} |\ell_j(x)|$$

λ is called the Lebesgue function, and Λ the Lebesgue constant.

Lebesgue's Lemma

With the operator norm of interpolation, here is a classical result quantifying the quality of polynomial interpolation:

Lemma (Lebesgue)

Let $f \in C([a,b])$, and assume that $x_1, \ldots, x_{n+1}$ are distinct nodes on $[a,b]$. Then

$$\|f - I_n f\|_\infty \leqslant (1 + \Lambda) \inf_{p \in P_n} \|f - p\|_\infty$$

Thus, the Lebesgue constant governs the accuracy of polynomial interpolants.

Nodal sets

This raises the question: can we minimize Lebesgue constants?

Let Λ_n denote the Lebesgue constant for $n + 1$ points.

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- For *any* triangular array of points, $\Lambda_n \stackrel{n\uparrow\infty}{\longrightarrow} \infty$.
- For equidistant nodes, $\Lambda_n \sim 2^n$. (This is bad!)
- For “Chebyshev" nodes , $\Lambda_n \sim \log n$. (This is asymptotically optimal)

Chebyshev nodes x_j (say on $[-1,1]$) are equidistant nodes under the cosine map:

$$x_j = \cos\theta_j, \qquad \theta_j = \frac{2j-1}{2n+2}\pi, \qquad j = 1, \ldots, n+1.$$

Another error estimate

There are alternative strategies to computing error estimates for interpolation.

With $x_1, \ldots, x_{n+1}$ the nodes, the *nodal polynomial* $\omega(x)$ is

$$\omega(x) := \prod_{j=1}^{n+1} (x - x_j).$$

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This can be used to construct an error estimate:

Theorem

Let $x_1, \ldots, x_{n+1}$ be distinct nodes on $[a, b]$. Then

$$|f(x) - [I_n f](x)| = \frac{\left|f^{(n+1)}(\xi)\right|}{(n+1)!} |\omega(x)|,$$

where $\xi = \xi(x)$ lies in $[a, b]$.