Polynomial approximation, I

MATH 6610 Lecture 25

November 11, 2020

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Like with Fourier Series, there is a "completeness" statement, ensuring that polynomials have sufficient approximation capacity for continuous functions.

Theorem (Weierstrass)

Let $f : [a, b] \to \mathbb{R}$ *be continuous with* $b - a < \infty$. *Then there exists a sequence of polynomials* $\{p_n\}_{n=0}^{\infty}$, *with* $\deg p_n \leqslant n$, *such that*

$$\lim_{n \uparrow \infty} \sup_{x \in [a,b]} |f(x) - p_n(x)| = 0.$$

This Weierstrass approximation result ensures that it's possible to construct accurate approximating polynomials.

Polynomial interpolation

How do we construct polynomial approximations? Interpolation is the simplest strategy.

Given a continuous function $f : [a, b] \to \mathbb{R}$, we seek an interpolant from the space

$$P_n := \text{span}\{1, x, \ldots, x^n\}$$

on the points $\{x_1, \ldots x_{n+1}\} \subset [a, b]$.

constraints. degrees of freedom

Generate $p \in P_n$ s.t. $p(x_j) = f(x_j) \quad \forall j = 1, \ldots n+1$

$$p(x) = \sum_{k=0}^{n} c_{k+1} x^k$$

$$p(x_j) = f(x_j) \implies \sum_{k=0}^{n} c_{k+1} x_j^k = f(x_j)$$

$\downarrow$

linear condition on $\{c_k\}$

constraints:

$$\begin{pmatrix} 1 & x_1^1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2^1 & x_2^2 & \cdots & x_2^n \\ \vdots & & \ddots & & \vdots \\ 1 & x_{n+1}^1 & x_{n+1}^2 & \cdots & x_{n+1}^n \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n+1} \end{pmatrix} = \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_{n+1}) \end{pmatrix}$$

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This results in a linear algebra problem: given $p \in P_n$, then

$$p(x) = \sum_{j=0}^{n} c_j x^j,$$

for some cofficients c_j. By enforcing equality on the interpolation grid:

$$p(x_j) = f(x_j), \qquad\qquad j = 1, \ldots, n+1,$$

we obtain a linear system for the coefficients c_j:

$$Vc = f, \qquad\qquad (V)_{j,\ell} = x_j^{\ell-1}, \qquad 1 \leq j, \ell \leq n+1$$

and $f_j = f(x_j)$, $c_j = c_j$.

Interpolation existence and uniqueness

$$Vc = f, \qquad\qquad (V)_{j,\ell} = x_j^{\ell-1},$$

V is called a *Vandermonde matrix*.

Existence and uniqueness of the interpolation problem is equivalent to that of the linear system.

Theorem (Interpolation unisolvence)

If the nodes $\{x_1, \ldots, x_{n+1}\} \subset [a, b]$ are all distinct, then given any continuous f, there is a unique degree-n polynomial that interpolates f on these nodes.

(not true as written in more than 1 dimension)

Interpolation existence and uniqueness

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There are (at least) two ways to prove this:

- Show that $\det V \neq 0$.
- Explicitly construct an interpolant (non-linear-algebraic).

Proof: Uniqueness: assume p, q are both degree-n interpolants.

$$p(x_j) = q(x_j), \quad j = 1 \ldots n+1$$

$$x_j \neq x_k \quad \forall \, j \neq k$$

$$\Longrightarrow p(x) - q(x) \in P_n$$

$p - q$ has $n+1$ zeros @ $x_1 \ldots x_{n+1}$.

$\Longrightarrow$ Fund. theorem of algebra implies $p - q \equiv 0$.

Existence

[Strategy A) Show $\det V \neq 0$. (by induction).

Lemma: If V_n is the $n \times n$ Vandermonde matrix:

$$V_n = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & \vdots & \ddots & \vdots \\ \vdots & \vdots & & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix}$$

Then: $\det V_n = \prod_{1 \le j < k \le n} (x_k - x_j)$.

<u>Proof of lemma</u>: (induction)

<u>$n=1$</u>: $V_1 = (1)$, $\det V_1 = \prod_{1 \le j < k \le 1} (x_k - x_j)$

$= 1$

<u>$n=2$</u>: $V_2 = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix}$, $\det V_2 = x_2 - x_1$

$= \prod_{1 \le j < k \le 2} (x_k - x_j)$

inductive step: assume $\det V_n = \prod_{1 \le j < k \le n} (x_k - x_j)$

$$V_{n+1} = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^n \end{pmatrix}$$

$$\det V_{n+1} = \sum_{l=1}^{n+1} (-1)^{n+l-1} x_{n+1}^{l-1} M_l$$

Laplace exp along last row

determinant minor

M_l = determinant of V_{n+1} w/o row $n+1$ and column l.

Note M_l does <u>not</u> depend on x_{n+1}.

$$M_l = M_l(x_1 \ldots x_n)$$

$\Rightarrow$ $\det V_{n+1}$, as a function of x_{n+1}, is an element of P_n.

As a function of x_{n+1}, then $\det V_{n+1}$:

- has roots @ $x = x_1 \ldots x_n$.
  (replacing x_{n+1} by x_j makes row $n+1$ of V_{n+1} identical to row $j \Rightarrow \det V_{n+1} = 0$).

- leading coefficient of $\det V_{n+1}$ is $\det V_n$.

$$\det V_{n+1} = \sum_{\ell=1}^{n+1} (-1)^{n+\ell-1} x_{n+1}^{\ell-1} M_\ell.$$

$$= (-1)^{2n} x_{n+1}^n M_{n+1} + (\text{lower order terms})$$

$$= x_{n+1}^n M_{n+1} + \cdots$$

determinant of V_{n+1} with row $n+1$, column $n+1$ removed

$$= x_{n+1}^n (\det V_n) + \cdots$$

$$\Longrightarrow \det V_{n+1} = C \cdot \prod_{\ell=1}^{n} (x_{n+1} - x_\ell)$$

$$= (\det V_n) \prod_{\ell=1}^{n} (x_{n+1} - x_\ell)$$

$$= \left(\prod_{1 \le j < k \le n} (x_k - x_j) \right) \prod_{\ell=1}^{n} (x_{n+1} - x_\ell)$$

$$= \prod_{1 \le j < k \le n+1} (x_k - x_j). \quad \blacksquare$$

So: $\det V_n = \displaystyle\prod_{1 \le j < k \le n} (x_k - x_j) \neq 0$ if $x_k \neq x_j$ $\forall\, k \neq j$.

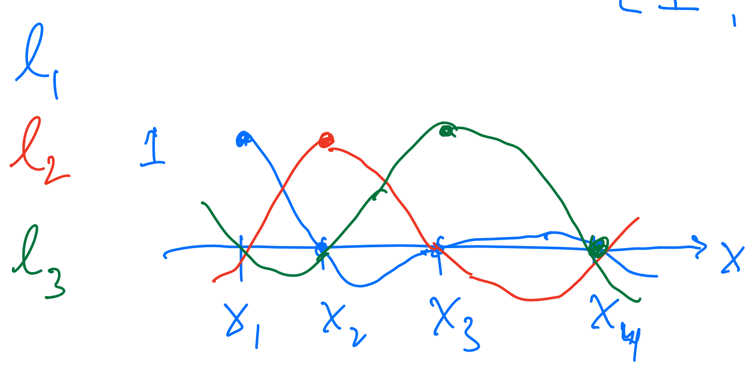
$\Longrightarrow$ $Vc = f$ has a (unique) solution.

<u>Strategy B</u> (existence)

Define $$\ell_j(x) = \prod_{\substack{k=1 \dots n \\ k \neq j}} \frac{(x - x_k)}{(x_j - x_k)} \in P_{n-1}$$

$(j = 1 \dots n)$

Note: $$\ell_j(x_k) = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$



Then: $\sum_{j=1}^{n} f(x_j)\, \ell_j(x) \in P_{n-1}$

satisfies $\sum_{j=1}^{n} f(x_j)\, \ell_j(x_k)$

$$= \sum_{j=1}^{n} f(x_j)\, \delta_{j,k} = f(x_k) \quad \blacksquare$$

The functions $\ell_j(x)$ are called cardinal Lagrange functions.