Polynomial approximation, I

MATH 6610 Lecture 25

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Like with Fourier Series, there is a "completeness" statement, ensuring that polynomials have sufficient approximation capacity for continuous functions.

Theorem (Weierstrass)

Let $f : [a, b] \to \mathbb{R}$ *be continuous with* $b - a < \infty$*. Then there exists a sequence of polynomials* $\{p_n\}_{n=0}^{\infty}$*, with* $\deg p_n \leqslant n$*, such that*

$$\lim_{n \uparrow \infty} \sup_{x \in [a,b]} |f(x) - p_n(x)| = 0.$$

This Weierstrass approximation result ensures that it's possible to construct accurate approximating polynomials.

Polynomial interpolation

How do we construct polynomial approximations? Interpolation is the simplest strategy.

Given a continuous function $f : [a, b] \to \mathbb{R}$, we seek an interpolant from the space

$$P_n := \mathrm{span}\{1, x, \ldots, x^n\}$$

on the points $\{x_1, \ldots x_{n+1}\} \subset [a, b]$.

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This results in a linear algebra problem: given $p \in P_n$, then

$$p(x) = \sum_{j=0}^{n} c_j x^j,$$

for some cofficients c_j. By enforcing equality on the interpolation grid:

$$p(x_j) = f(x_j), \qquad j = 1, \ldots, n + 1,$$

we obtain a linear system for the coefficients c_j:

$$V\boldsymbol{c} = \boldsymbol{f}, \qquad (V)_{j,\ell} = x_j^{\ell-1},$$

and $f_j = f(x_j)$, $c_j = c_j$.

Interpolation existence and uniqueness

$$Vc = f, \qquad\qquad (V)_{j,\ell} = x_j^{\ell-1},$$

V is called a *Vandermonde matrix*.

Existence and uniqueness of the interpolation problem is equivalent to that of the linear system.

Theorem (Interpolation unisolvence)

If the nodes $\{x_1, \ldots, x_{n+1}\} \subset [a, b]$ *are all distinct, then given any continuous* f*, there is a unique degree-*n *polynomial that interpolates* f *on these nodes.*

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There are (at least) two ways to prove this:

- Show that $\det V \neq 0$.
- Explicitly construct an interpolant (non-linear-algebraic).