Fourier approximation

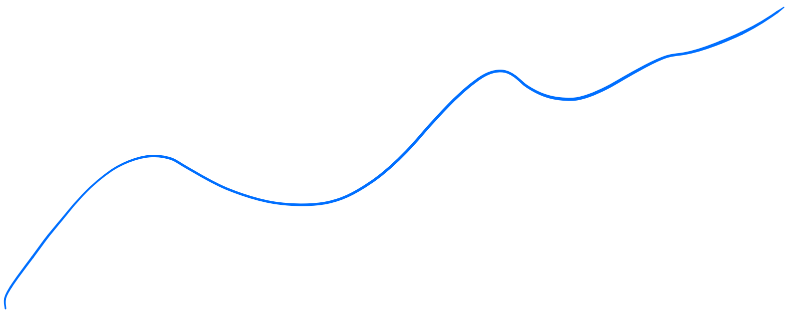
MATH 6610 Lecture 24

November 9, 2020

(Applied) Approximation theory

The last batch of material in this class regards approximation theory and methods:

How can we represent (generally infinite-dimensional) functions with finite data on a computer?



(Applied) Approximation theory

The last batch of material in this class regards approximation theory and methods:

How can we represent (generally infinite-dimensional) functions with finite data on a computer?

We'll investigate three geeneral strategies for approximation:

- Fourier approximation
- polynomial approximation (most of the time)
- rational approximation

For simplicity, we'll consider only scalar-valued functions of one variable.

$f: \mathbb{C} \to \mathbb{C}$

$f: \mathbb{R} \to \mathbb{R}$

Major questions

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- When are infinite-dimensional functions efficiently representable in finite-dimensional spaces?
- Are (near-)optimal finite-dimensional representations computable?
- In 6620: using approximation techniques to solve differential equations

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With Fourier approximation today, we'll address the first problem with the following theme:

$$\text{Smoothness} \Longrightarrow \text{Compressibility}$$

smooth graph

accurate representation with finite data.

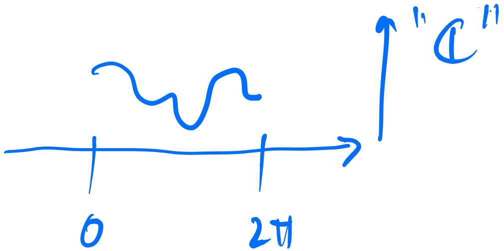
Fourier Series

We'll consider functions in a Hilbert space: (of scalar-valued functions)

$$L^2 = L^2([0, 2\pi]; \mathbb{C}) = \left\{ f : [0, 2\pi] \to \mathbb{C} \;\middle|\; \|f\| < \infty \right\},$$

metrized with the norm,

$$\|f\|^2 := \|f\|_{L^2}^2 = \langle f, f \rangle, \qquad\qquad \langle f, g \rangle := \int_0^{2\pi} f(x) g(x)^* \mathrm{d}x$$



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Fact: the set of functions $\left\{ e^{ikx} \right\}_{k \in \mathbb{Z}} \subset L^2$ is complete in L^2.

Theorem (L^2 completeness of Fourier Series)

For any $f \in L^2$, there exists a sequence $\left\{ \widehat{f}_k \right\}_{k \in \mathbb{Z}} \subset \mathbb{C}$ such that

L^2 norm

$$\lim_{n \uparrow \infty} \left\| f - \sum_{|k| \leqslant n} \widehat{f}_k e^{ikx} \right\| = 0.$$

Fourier projections, I

The coefficient sequence $\widehat{f}_k$ even has an explicit formula.

By defining an orthonormal basis,

$\wedge$

L^2-

$$\phi_k(x) := \frac{1}{\sqrt{2\pi}} e^{ikx}, \qquad k \in \mathbb{Z}$$

$$\langle \phi_k, \phi_\ell \rangle = \delta_{k,\ell}$$

then the $\widehat{f}_k$ coefficients are defined by a linear-algebraic-like projection:

$$\widehat{f}_k := \langle f, \phi_k \rangle . = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x) e^{-ikx}\, dx$$

Fourier projections, II

Furthermore, these coefficients define an optimal approximation.

Theorem

Given $f \in L^2$, the optimal trigonometric polynomial f_n of frequency at most n is given by:

$$f_n := \underset{g \in V_n}{\arg\min} \|f - g\|, \qquad f_n(x) = \sum_{|k| \leqslant n} \widehat{f}_k \phi_k(x),$$

$\hat{f}_k = \langle f, \varphi_k \rangle.$

where $V_n := \operatorname{span}\{\phi_k\}_{|k| \leqslant n}$.

In fact, the map $f \mapsto f_n$ is a linear orthogonal projection.

From LA: $v \in \mathbb{C}^n$, $\{q_j\}_{j=1}^r$ is ~~a~~ set of orthonormal vectors.

Then $\mathbb{C}^n$-orthogonal projection of v onto $\operatorname{span}\{q_j\}_{j=1}^r$ is

$QQ^* v$, $Q = \begin{pmatrix} | & & | \\ q_1 & \cdots & q_r \\ | & & | \end{pmatrix}$. $QQ^* v = \sum_{j=1}^{r} \langle v, q_j \rangle q_j$

$$f_n = \sum_{|k|\le n} \hat{f}_k \varphi_k = \sum_{|k|\le n} \langle f, \varphi_k \rangle \varphi_k$$

$P_n f := f_n$, $P_n : L^2 \longrightarrow V_n$

P_n is a linear orthogonal projector.

$\mathrm{range}(P_n) \perp \ker(P_n)$

$\mathrm{range}(P_n) \oplus \ker(P_n) = L^2$

$P_n^2 f = P_n f$

Fourier projections, II

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$$f_n := \operatorname*{arg\,min}_{g \in V_n} \|f - g\|, \qquad f_n(x) = \sum_{|k| \leqslant n} \widehat{f}_k \phi_k(x),$$

where $V_n := \operatorname{span}\{\phi_k\}_{|k| \leqslant n}$.

$\hat{f}_k = \langle f, \phi_k \rangle$

In fact, the map $f \mapsto f_n$ is a linear orthogonal projection.

A related fact is Parseval's identity:

$$\|f\|^2 = \sum_{k \in \mathbb{Z}} \left| \widehat{f}_k \right|^2.$$

Efficiency of Fourier representations

Given $f \in L^2$, the $(2n+1)$-term approximation,

$$f_n(x) = \sum_{|k| \leqslant n} \widehat{f}_k \phi_k(x), \qquad \widehat{f}_k := \langle f, \phi_k \rangle,$$

is an optimal approximation, and converges to f as $n \uparrow \infty$.

How *quickly* does this converge?

$f_n \simeq f \quad \left(n \uparrow \infty, \; f_n \longrightarrow f \right)$

$\left(f_n = \operatorname{argmin}_{g \in V_n} \|f-g\| \right)$

This is relevant for computational problems: instead of storing f, store $\{\hat{f}_k\}_{|k| \leq n}$.

Smoothness

Smoothness plays a central role in classical optimal approximation results:
The smoother the function f, the faster that f_n converges to f.

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One standard measure of smoothness: membership in a *Sobolev* space.

r'th derivative of f.

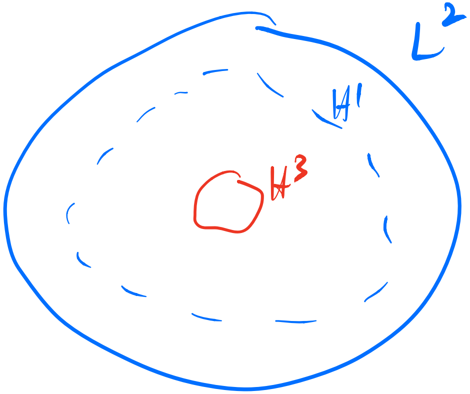
Given $s \in \{0, 1, 2 \ldots\}$

$$H^s := H^s_p([0,2\pi];\mathbb{C}) := \Big\{ f : [0,2\pi] \to \mathbb{C} \;\Big|\; \left\| f^{(r)} \right\|_{L^2} < \infty \text{ for } r = 0, 1, \ldots, s$$
$$\text{and } f^{(r)}(0) = f^{(r)}(2\pi) \text{ for } r = 0, 1, \ldots, s-1 \Big\},$$

periodic

metrized with the norm $\|f\|^2_{H^s} := \sum_{r=0}^{s} \left\| f^{(r)} \right\|^2_{L^2}$.

$(H^0 = L^2)$



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The smoother the function f, the faster that f_n converges to f.

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metrized with the norm $\|f\|^2_{H^s} := \sum_{r=0}^{s} \left\| f^{(r)} \right\|^2_{L^2}$.

The rate of convergence of Fourier approximations depends on the smoothness parameter s.

$\sum_{|k| \leq n} \hat{f}_k \phi_k(x)$

Theorem

Assume $f \in H^s$ for some $s \geqslant 0$. Then the $(2n+1)$-term approximation f_n commits the error

$$\|f - f_n\|_{L^2} \leqslant n^{-s} \|f\|_{H^s}.$$

optimal approximation

$n=100$, $s=1 \Rightarrow \|f-f_n\|_{L^2} \lesssim \frac{1}{100}$ ("first-order" convergence)

$s=1$

$n=100$, $s=10 \Rightarrow \|f-f_n\|_{L^2} \lesssim 10^{-20}$

"high-order" convergence
$(s \gg 1)$

smoothness $\Longrightarrow$ high-order convergence.

s: rate of convergence (n^{-s})

Proof: $f \in H^s$

Assume $s \geq 0$. If not: $\|f_n - f\|_{L^2} \leq \|f\|_{L^2}$

But $f - f_n = \sum_{|j|>n} \hat{f}_j \varphi_j(x)$

$\sum_j \hat{f}_j \varphi_j$

Parseval

$\Longrightarrow \|f-f_n\|^2 = \sum_{|j|>n} |\hat{f}_j|^2$

$\leq \sum_{j \in \mathbb{Z}} |\hat{f}_j|^2 = \|f\|_{L^2}^2$

If $s>0$: $f^{(r)} \in L^2$, $0 \leq r \leq s$

$f^{(r)} \in L^2 \Longrightarrow f^{(r)}(x) = \sum_{j \in \mathbb{Z}} \hat{f}_j^{(r)} \varphi_j(x)$

$\hat{f}_j^{(r)} = \langle f^{(r)}, \varphi_j \rangle$

(I.e. $f^{(r)}$ has a Fourier Series)

$$\hat{f}_j = \langle f, \varphi_j \rangle = \int_0^{2\pi} f(x)\, \varphi_j^*(x)\, dx$$

$$= \frac{1}{-ij} \int_0^{2\pi} f(x)\, (-ij)\, \varphi_j^*(x)\, dx \quad (j \neq 0)$$

Note: $\varphi_j^*(x) = \frac{1}{\sqrt{2\pi}} e^{-ijx}$

$$\Longrightarrow (-ij)\, \varphi_j^*(x) = \frac{d}{dx} \varphi_j^*(x)$$

So: $\hat{f}_j = \langle f, \varphi_j \rangle = \frac{1}{-ij} \int_0^{2\pi} f(x)\, (\varphi_j^*(x))'\, dx$

$$= \frac{1}{-ij} \left[\cancel{f(x)\, \varphi_j^*(x) \Big|_0^{2\pi}}^{\,0} - \int_0^{2\pi} f'(x)\, \varphi_j^*(x)\, dx \right]$$

$$= \frac{1}{ij} \int_0^{2\pi} f'(x)\, \varphi_j^*(x)\, dx$$

$$= \frac{1}{ij} \hat{f}_j^{(1)}$$

In general: $\hat{f}_j^{(r)} = \frac{1}{ij} \hat{f}_j^{(r+1)}$ if $r < s$

$\rightarrow \hat{f}_j^{(r)} = \int_0^{2\pi} f^{(r)}(x)\, \varphi_j^*(x)\, dx$

$$= \frac{1}{-ij} \int_0^{2\pi} f^{(r)}(x) \left(\varphi_j^*(x)\right)' dx$$

0 since $f \in H^s$, $s > r$

$$= -\frac{1}{ij} \left[\cancel{f^{(r)}(x) \varphi_j^*(x) \Big|_0^{2\pi}} - \int_0^{2\pi} f^{(r+1)}(x) \varphi_j^*(x) dx \right]$$

$$= \frac{1}{ij} \int_0^{2\pi} f^{(r+1)} \varphi_j^*(x) dx = \frac{1}{ij} \hat{f}_j^{(r+1)}$$

$$\hat{f}_j^{(r)} = \frac{1}{ij} \hat{f}_j^{(r+1)} \qquad 0 \le r \le s-1$$

$$\Downarrow$$

$$\hat{f}_j = \left(\frac{1}{ij}\right)^s \hat{f}_j^{(s)}$$

$$\|f - f_n\|_{L^2}^2 = \Big\| \sum_{j \in \mathbb{Z}} \hat{f}_j \varphi_j(x) - \sum_{|j| \le n} \hat{f}_j \varphi_j(x) \Big\|_{L^2}^2$$

$$= \Big\| \sum_{|j| > n} \hat{f} \varphi_j(x) \Big\|_{L^2}^2$$

Parseval

$$= \sum_{|j|>n} |\hat{f}_j|^2$$

$$= \sum_{|j|>n} \frac{1}{|j|^{2s}} |\hat{f}_j^{(s)}|^2$$

$$\leq \sum_{|j|>n} \frac{1}{n^{2s}} |\hat{f}^{(s)}_j|^2 \quad (|j| > n)$$

$$= \frac{1}{n^{2s}} \sum_{|j|>n} |\hat{f}^{(s)}_j|^2 \leq \frac{1}{n^{2s}} \sum_{j\in\mathbb{Z}} |\hat{f}^{(s)}_j|^2$$

$$= \frac{1}{n^{2s}} \|f^{(s)}\|_{L^2}^2$$

$$\leq \frac{1}{n^{2s}} \|f\|_{H^s}^2$$

def'n of H^s

$$\Longrightarrow \|f - f_n\|_{L^2}^2 \leq n^{-2s} \|f\|_{H^s}^2 \qquad \blacksquare$$