Fourier approximation

MATH 6610 Lecture 24

November 9, 2020

(Applied) Approximation theory

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How can we represent (generally infinite-dimensional) functions with finite data on a computer?

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How can we represent (generally infinite-dimensional) functions with finite data on a computer?

We'll investigate three geeneral strategies for approximation:

- Fourier approximation
- polyomial approximation
- rational approximation

For simplicity, we'll consider only scalar-valued functions of one variable.

Major questions

We'll be concerned with the following questions:

- When are infinite-dimensional functions efficiently representable in finite-dimensional spaces?
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- In 6620: using approximation techniques to solve differential equations

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With Fourier approximation today, we'll address the first problem with the following theme:

$$\text{Smoothness} \Longrightarrow \text{Compressibility}$$

Fourier Series

We'll consider functions in a Hilbert space:

$$L^2 = L^2([0, 2\pi); \mathbb{C}) = \left\{ f : [0, 2\pi] \to \mathbb{C} \;\middle|\; \|f\| < \infty \right\},$$

metrized with the norm,

$$\|f\| := \|f\|_{L^2} = \langle f, f \rangle, \qquad\qquad \langle f, g \rangle := \int_0^{2\pi} f(x) g(x)^* \mathrm{d}x$$

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Fact: the set of functions $\left\{e^{ikx}\right\}_{k \in \mathbb{Z}} \subset L^2$ is complete in L^2.

Theorem (L^2 completeness of Fourier Series)

For any $f \in L^2$, there exists a sequence $\left\{\widehat{f}_k\right\}_{k \in \mathbb{Z}} \subset \mathbb{C}$ such that

$$\lim_{n \uparrow \infty} \left\| f - \sum_{|k| \leqslant n} \widehat{f}_k e^{ikx} \right\| = 0.$$

Fourier projections, I

The coefficient sequence $\widehat{f}_k$ even has an explicit formula.

By defining an orthonormal basis,

$$\phi_k(x) := \frac{1}{\sqrt{2\pi}} e^{ikx},$$

then the $\widehat{f}_k$ coefficients are defined by a linear-algebraic-like projection:

$$\widehat{f}_k := \langle f, \phi_k \rangle .$$

Fourier projections, II

Furthermore, these coefficients define an optimal approximation.

Theorem

Given $f \in L^2$*, the optimal trigonometric polynomial* f_n *of frequency at most* n *is given by:*

$$f_n := \underset{g \in V_n}{\arg\min} \|f - g\|, \qquad f_n(x) = \sum_{|k| \leqslant n} \widehat{f}_k \phi_k(x),$$

where $V_n := \mathrm{span}\, \{\phi_k\}_{|k| \leqslant n}$.

In fact, the map $f \mapsto f_n$ is a linear orthogonal projection.

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A related fact is Parseval's identity:

$$\|f\|^2 = \sum_{k \in \mathbb{Z}} \left|\widehat{f}_k\right|^2.$$

Efficiency of Fourier representations

Given $f \in L^2$, the $(2n+1)$-term approximation,

$$f_n(x) = \sum_{|k| \leqslant n} \widehat{f}_k \phi_k(x), \qquad \widehat{f}_k := \left\langle f, \phi_k \right\rangle,$$

is an optimal approximation, and converges to f as $n \uparrow \infty$.

How *quickly* does this converge?

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Smoothness plays a central role in classical optimal approximation results: The smoother the function f, the faster that f_n converges to f.

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One standard measure of smoothness: membership in a *Sobolev* space.

$$H^s := H_p^s([0, 2\pi]; \mathbb{C}) := \left\{ f : [0, 2\pi] \to \mathbb{C} \;\middle|\; \left\| f^{(r)} \right\|_{L^2} < \infty \text{ for } r = 0, 1, \ldots, s \right.$$
$$\left. \text{and } f^{(r)}(0) = f^{(r)}(2\pi) \text{ for } r = 0, 1, \ldots, s-1 \right\},$$

metrized with the norm $\|f\|_{H^s}^2 := \sum_{r=0}^{s} \left\| f^{(r)} \right\|_{L^2}^2$.

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metrized with the norm $\|f\|^2_{H^s} := \sum_{r=0}^{s} \left\| f^{(r)} \right\|^2_{L^2}$.

The rate of convergence of Fourier approximations depends on the smoothness parameter s.

Theorem

Assume $f \in H^s$ for some $s \geqslant 0$. Then the $(2n+1)$-term approximation f_n commits the error

$$\|f - f_n\|_{L^2} \leqslant n^{-s} \|f\|_{H^s}.$$