# Iterative methods (for linear systems)

MATH 6610 Lecture 22

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Trefethen & Bau: Lectures 32, 35, 38

## Direct methods

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- orthogonalize vectors
- solve linear systems
- compute eigenvalues

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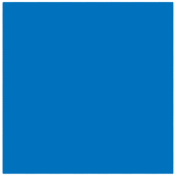
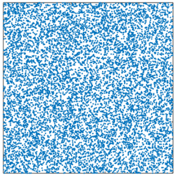
For very large matrices, e.g., $A \in \mathbb{C}^{10^6 \times 10^6}$, such procedures are not practical.

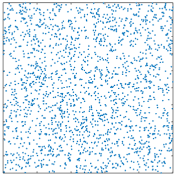
For such large matrices of general type, there is not much that can be done.

But in practice, matrices are *sparse*, having a reasonably small percentage of nonzero entries.


Dense matrix

10% density


1% density


0.2% density

## Direct methods and sparse matrices

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Unfortunately, direct methods operating on sparse matrix frequently result in dense matrices:

- QR factorizations
- LU factorizations
- Eigenvalue algorithms

Even just storing such matrices can be impossible in practice.

## Iterative methods

In contrast to direct methods, iterative methods produce answers that gradually approach the solution, performing only operations that generally don't require large dense matrices.

There are two major problems that iterative methods generally seek to solve:

- Solve linear systems
- Compute eigenvalues/eigenvectors

We'll briefly discuss the general ideas for the first class of problems.

## Themes of iterative methods

$$Ax = b$$

Many iterative methods focus on ensuring the following properties

- A sequence of solution approximations, $x_0, x_1, x_2, \ldots$ is constructed, with a new approximation formed at every iteration.
- The update $x_k \mapsto x_{k+1}$ typically involves only efficient matrix-vector multiplications (e.g., exploiting sparsity)
- The sequence $\{x_k\}$ gradually approaches the solution as $k \uparrow \infty$.
- Termination is frequently judged by inspecting the residual $\|Ax_k - b\|$.

## Stationary iterative methods

$$Ax = b$$

Stationary iterative methods (also “relaxation methods") are among the first types of iterative methods developed.

The basic idea of stationary iterative methods is as follows: we update solutions linearly:

$$Bx_{k+1} = Cx_k + d,$$

where $B$ and $C$ are matrices, and $d$ is a vector.

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where $B$ and $C$ are matrices, and $d$ is a vector. In order to prove convergence, we attempt to choose $B$, $C$, and $d$ so that

$$e_{k+1} = Fe_k, \qquad\qquad e_k := x_k - x$$

We can ensure this relation if we make the choice,

$$B - C = A, \qquad\qquad d = b.$$

## Stationary iterative methods and convergence

$$e_{k+1} = F e_k, \qquad\qquad e_k := x_k - x$$

With the above relation, then we obtain convergence if

$$\lim_{k \to \infty} F^k = 0.$$

Ensuring this condition typically depends on the matrix $A$ and on what kind of decomposition $A = B - C$ is chosen.

## Some choices

$$Bx_{k+1} = Cx_k + b, \qquad\qquad A = B - C.$$

Suppose that $A = L + D + U$ where

- $L$ and $U$ are the lower- and upper-triangular portions of $A$, respectively,
- $D$ is the diagonal portion of $A$.

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- Jacobi method: $B = D$, $C = -L - U$.
- Gauss-Seidel method: $B = D + L$, $C = -U$.
- Successive over-relaxation method: $B = D + \alpha L$, $C = (1 - \alpha)D - \alpha U$.

These methods have strong theory when applied to discretizations of Laplace's equation.

## Krylov subspace methods

Krylov subspace methods are a general class of iterative methods.

Krylov subspace methods build approximations from the subspace,

$$\operatorname{span}\left\{b, Ab, A^2 b, \ldots, A^{r-1} b\right\},$$

where $r$ is either adaptively or *a priori* specified.

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Of course there are some implementation details:

- Neither $A^k$ nor $A^k b$ are ever explicitly computed.
- Some orthogonalization is typically performed to avoid ill-conditioning.

## Some Krylov subspace methods

Krylov subspace methods are among the most popular and effective iterative methods.

For solving linear systems, examples are

- Conjugate Gradient (CG) and variants (CGStab, BiCGStab, etc.)
- Generalized minimum residuals (GMRES) and variants (MINRES, QMR, etc.)

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For computing eigenvalues some iterative methods are

- Lanczos iteration (for Hermitian $A$)
- Arnoldi iteration (for general $A$)

## Preconditioning

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Typically, $P$ is chosen as an "approximate" inverse of $A$, such as its diagonal.

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Effective preconditioning serves both to stabilize numerical computations, and to accelerate iterative methods (result in fewer iterations).