Eigenvalue algorithms: The QR algorithm

MATH 6610 Lecture 20

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Trefethen & Bau: Lecture 28

Simultaneous power iteration, I

Let $(\lambda_j, v_j)_{j=1}^{n}$ be the ordered eigenpairs of A, with $|\lambda_j| > |\lambda_{j+1}|$.

As relatively ineffective as power iteration is, consider applying it to 2 vectors v, w, which have expansions

$$v = \sum_{j=1}^{n} c_j v_j, \qquad w = \sum_{j=1}^{N} d_j v_j.$$

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$$v = \sum_{j=1}^{n} c_j v_j, \qquad\qquad w = \sum_{j=1}^{N} d_j v_j.$$

For large k, then:

$$\begin{aligned} A^k \left[v \;\; w \right] &\approx \begin{pmatrix} \lambda_1^k c_1 v_1 & \lambda_1^k \left[d_1 v_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^k v_2 \right] \end{pmatrix} \\ &= (v_1 \;\; v_2) \begin{pmatrix} c_1 \lambda_1^k & d_1 \lambda_1^k \\ 0 & d_2 \left(\frac{\lambda_2}{\lambda_1}\right)^k \end{pmatrix} =: QR \end{aligned}$$

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Initialize $Q_0^{PI} = I$. For $k = 0, 1, \ldots,$

1. $A_{k+1}^{PI} := AQ_k^{PI}$
2. $Q_{k+1}^{PI} R_{k+1}^{PI} := A_{k+1}^{PI}$

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For large k, we expect $Q_k^{PI} \to V$. In fact, we can show that if we have the QR decomposition $A^k = Q_k R_k$, then

$$Q_k^{PI} R_k^{PI} R_{k-1}^{PI} \cdots R_1^{PI} = Q_k R_k$$

So simultaneous power iteration compute Q_k implicitly.

The QR algorithm

The QR algorithm is a procedure for computing eigenvalues.

(It is distinct from the QR decomposition, but does use QR decompositions.)

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At face value, it's remarkable that this algorithm does *anything* useful.

In fact, this actually is performing simultaneous power iteration in disguise.

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This last relation yields the following result via induction:

$$A^k = \left(Q_1^{QR} Q_2^{QR} \cdots Q_k^{QR} \right) \left(R_k^{QR} R_{k-1}^{QR} \cdots R_1^{QR} \right)$$

The QR algorithm's QR decomposition

Finally, we can uncover what the QR algorithm is doing since we have uncovered two QR decompositions of A:

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Therefore: with $A = V \Lambda V^*$ the eigenvalue decomposition of A, then:

$$\left(Q_1^{QR} Q_2^{QR} \cdots Q_k^{QR}\right) = Q_k^{PI} \xrightarrow{k \uparrow \infty} V.$$

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Great, but the QR algorithm doesn't compute the entire matrix,

$$\left(Q_1^{QR} Q_2^{QR} \cdots Q_k^{QR}\right),$$

it just computes Q_k^{QR}.

The QR algorithm computes eigenvalues

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The final property to note is that A_k^{QR} is unitarily equivalent to A:

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But we know that $\left(Q_1^{QR} Q_2^{QR} \cdots Q_k^{QR}\right) \rightarrow V$. Therefore:

$$\begin{aligned} A_k^{QR} &= \left(Q_1^{QR} Q_2^{QR} \cdots Q_k^{QR}\right)^* \; A \; \left(Q_1^{QR} Q_2^{QR} \cdots Q_k^{QR}\right) \\ &\stackrel{k\uparrow\infty}{\longrightarrow} V^* A V = \Lambda. \end{aligned}$$

The QR algorithm and convergence

The QR algorithm therefore performs an eigenvalue decomposition. For real, symmetric matrices, we have convergence

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In fact: the real symmetric assumption is not necessary. For (fairly) general matrices A, the QR algorithm computes A_k^{QR} that converges to the Schur factor T in the Schur decomposition

$$A = QTQ^*.$$