Eigenvalue algorithms: Rayleigh iteration

MATH 6610 Lecture 19

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Trefethen & Bau: Lecture 27

Eigenvalues: power iteration

Let $A \in \mathbb{C}^{n \times n}$ be Hermitian with eigenpairs $(\lambda_j, v_j)_{j=1}^n$ that are simple and ordered such that $|\lambda_j| > |\lambda_{j+1}|$.

Power iteration performs the following basic steps:

Initialize with a vector v (e.g., randomly):

1. $v \leftarrow \frac{Av}{\|Av\|_2}$
2. $\lambda \leftarrow R_A(v)$
3. Return to step 1

For a large number of iterations, then (λ, v) converges to (λ_1, v_1).

To make this practically useful, need to supplement with termination criterion, deflation.

Power iteration's drawback

One major drawback of power iteration is that the ratio

$$r = \left| \frac{\lambda_{j+1}}{\lambda_j} \right|,$$

Recall convergence $\mathcal{O}(r^k)$ @ iteration k.

dictates the rate of convergence for computing eigenpair j.

This implies, in particular, that the effectiveness of this algorithm depends heavily on A.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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But power iteration is simple and the idea is attractive. Is there a way to accelerate power iteration?

Inverse iteration

Inverse iteration "modifies" the spectrum of A so that power iteration will be "more successful."

Spectrum has "larger" spacing.

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First we note that, given some $\mu \in \mathbb{C}$, the eigenvalues of $(A - \mu I)^{-1}$ are $1/(\lambda_j - \mu)$.

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The essential realization is that, if $\mu \approx \lambda_j$, then

$$\frac{1}{|\mu - \lambda_j|} \gg \frac{1}{|\mu - \lambda_k|}, \qquad k \neq j.$$

I.e., if we can somehow get a "reasonable" guess μ, then power iteration on $(A - \mu I)^{-1}$ will be very efficient.

Rayleigh iteration

The idea of Rayleigh iteration is to combine

- inverse iteration, which accelerates identification of eigenvectors, with
- Rayleigh quotient evaluations, which accelerate identification of eigenvalues.

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Initialize with a vector v and scalar μ (e.g., randomly):

1. $v \leftarrow (A - \mu I)^{-1} v$ (inverse iteration)
2. $v \leftarrow v / \|v\|_2$
3. ~~λ~~ μ $\leftarrow R_A(v)$ (Rayleigh quotient)
4. Return to step 1

In principle this is more expensive than power iteration: we must solve $(A - \mu I)x = v$ at *every* step.

Rayleigh iteration, II

The rather surprising fact: this is an *extremely* efficient algorithm.

Let $\lambda^{(k)}, v^{(k)}$ be the Rayleigh iteration eigenpair approximation at iteration k.
Let $\lambda^{(k+1)}, v^{(k+1)}$ be the Rayleigh iteration eigenpair approximation at iteration $k+1$.

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Theorem

For almost every initialization of Rayleigh iteration, there is some eigenpair (λ_j, v_j) of A such that

$$\left|\lambda_j - \lambda^{(k+1)}\right| \leqslant \left|\lambda_j - \lambda^{(k)}\right|^3, \qquad \left\|v_j - v^{(k+1)}\right\| \leqslant \left\|v_j - v^{(k)}\right\|^3$$

Convergence is <u>cubic</u>!

<u>Proof</u> (-ish): 1 exponent of convergence from power iteration
2 exponents from Rayleigh quotient.

Rayleigh iteration: choose initial vector v.

$v^{(0)} = v$

$\lambda^{(0)} = R_A(v)$

for $k=1,2,\ldots$

$$w = (A - \lambda^{(k-1)} I)^{-1} v^{(k-1)}$$

$$v^{(k)} = w / \|w\|$$

$$\lambda^{(k)} = R_A(v^{(k)})$$

before was μ

For HW: need termination + deflation

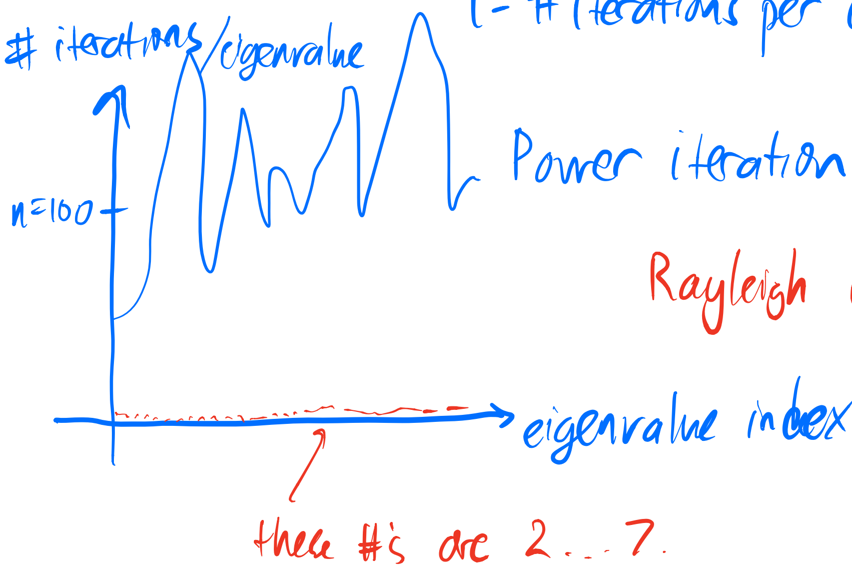
Rayleigh iteration is really efficient.

E.g. $A \in \mathbb{C}^{n \times n}$, $n=100$.

Power iteration: performs some iterations, periodically deflate.

↑ Same idea for Rayleigh iteration

Let's count # of iterations per deflation.



(= # iterations per eigenvalue found).

Power iteration (—)

Rayleigh iteration (- - -)

Hotelling deflation (A Hermitian)

$$A = \sum_{j=1}^{n} \lambda_j v_j v_j^* \quad \text{(E-V decomposition of } A\text{)}$$

Suppose find eigenpair (μ, w). [$= (\lambda_k, v_k)$ for some k]

$$A - \mu w w^* = \sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j v_j v_j^*$$

← rank-$(n-1)$ sum