Eigenvalue algorithms: Rayleigh iteration

MATH 6610 Lecture 19

October 21, 2020

Trefethen & Bau: Lecture 27

Eigenvalues: power iteration

Let $A \in \mathbb{C}^{n \times n}$ be Hermitian with eigenpairs $(\lambda_j, v_j)_{j=1}^n$ that are simple and ordered such that $|\lambda_j| > |\lambda_{j+1}|$.

Power iteration performs the following basic steps:

Initialize with a vector v (e.g., randomly):

1. $v \leftarrow \frac{Av}{\|Av\|_2}$
2. $\lambda \leftarrow R_A(v)$
3. Return to step 1

For a large number of iterations, then (λ, v) converges to (λ_1, v_1).

To make this practically useful, need to supplement with termination criterion, deflation.

Power iteration's drawback

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$$\left|\frac{\lambda_{j+1}}{\lambda_j}\right|,$$

dictates the rate of convergence for computing eigenpair j.

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But power iteration is simple and the idea is attractive. Is there a way to accelerate power iteration?

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The essential realization is that, if $\mu \approx \lambda_j$, then

$$\frac{1}{|\mu - \lambda_j|} \gg \frac{1}{|\mu - \lambda_k|}, \qquad k \neq j.$$

I.e., if we can somehow get a "reasonable" guess μ, then power iteration on $(A - \mu I)^{-1}$ will be very efficient.

Rayleigh iteration

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Initialize with a vector v and scalar μ (e.g., randomly):

1. $v \leftarrow (A - \mu I)^{-1} v$
2. $v \leftarrow v / \|v\|_2$
3. $\lambda \leftarrow R_A(v)$
4. Return to step 1

In principle this is more expensive than power iteration: we must solve $(A - \mu I)x = v$ at *every* step.

Rayleigh iteration, II

The rather surprising fact: this is an *extremely* efficient algorithm.

Let $\lambda^{(k)}, v^{(k)}$ be the Rayleigh iteration eigenpair approximation at iteration k.
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Theorem

For almost every initialization of Rayleigh iteration, there is some eigenpair (λ_j, v_j) *of* A *such that*

$$\left|\lambda_j - \lambda^{(k+1)}\right| \leqslant \left|\lambda_j - \lambda^{(k)}\right|^3, \qquad \left\|v_j - v^{(k+1)}\right\| \leqslant \left\|v_j - v^{(k)}\right\|^3$$

Convergence is <u>cubic</u>!