The Cholesky decomposition

MATH 6610 Lecture 17

October 16, 2020

Trefethen & Bau: Lecture 23

Hermitian positive-definite matrices

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First we note some properties of A:

- A is invertible
- The diagonal entries of A are real and strictly positive
- If $B \in \mathbb{C}^{m \times n}$ with $m \leqslant n$ is of full rank, then BAB^* is positive-definite

LU on positive-definite matrices

A general positive-definite matrix A has the form

$$A = \begin{pmatrix} a & - & v^* & - \\ | & & & \\ v & & A_2 & \\ | & & & \end{pmatrix}.$$

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Consider performing elimination on A:

$$A = L_1 B^* = \begin{pmatrix} 1 & - & 0 & - \\ | & & & \\ \frac{v}{a} & I & & \\ | & & & \end{pmatrix} \begin{pmatrix} a & - & v^* & - \\ | & & & \\ 0 & & A_2 - \frac{vv^*}{a} & \\ | & & & \end{pmatrix}$$

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i.e.,

$$A = L_1 \begin{pmatrix} a & - & 0 & - \\ | & & & \\ 0 & & A_2 - \frac{vv^*}{a} & \\ | & & & \end{pmatrix} L_1^* = \widetilde{L}_1 \begin{pmatrix} 1 & - & 0 & - \\ | & & & \\ 0 & & A_2 - \frac{vv^*}{a} & \\ | & & & \end{pmatrix} \widetilde{L}_1^*.$$

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Thus, we can repeat this process:

$$\begin{aligned} A &= \left(\widetilde{L}_1 \widetilde{L}_2 \cdots \widetilde{L}_{n-1}\right) \left(\widetilde{L}_1 \widetilde{L}_2 \cdots \widetilde{L}_{n-1}\right)^* \\ &=: LL^*. \end{aligned}$$

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Theorem

Every Hermitian positive definite matrix A has a unique symmetric LU, or Cholesky, decomposition: $A = LL^*$, where L is lower-triangular and invertible.

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One can perform symmetric pivoting on a Hermitian positive-definite matrix A: $A = PLL^*P^*$.

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However, pivoted Cholesky decompositions have another use:

Theorem

Every Hermitian positive semi-definite matrix A has a pivoted Cholesky decomposition: $A = PLL^*P^*$, where L is lower-triangular but need not invertible. This decomposition is in general not unique.