Pivoting and the LU factorization

MATH 6610 Lecture 16

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Trefethen & Bau: Lecture 21

LU and Gaussian elimination

Let $A \in \mathbb{C}^{n \times n}$ be an invertible matrix. Recall that, if Gaussian elimination succeeds, then

$$A = LU,$$

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"Standard" Gaussian elimination fails in some cases, e.g., with

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Pivoting, I

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This results in the decomposition,

$$A = P_1 L_1 \, P_2 L_2 \, \cdots \, P_{n-1} L_{n-1} U,$$

where P_j is a permutation matrix that permutes row j with row k for some $k \geqslant j$.

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One can show that $L_j P_k = P_k \widetilde{L}_j$ if $j < k$ for some other lower-triangular matrix $\widetilde{L}_j$, so that

$$A = \left(\prod_{j=1}^{n-1} P_j \right) \left(\prod_{j=1}^{n-1} \widetilde{L}_j \right) U.$$

Pivoted LU

In fact, we can show that this *row pivoting* strategy always works.

Theorem

If $A \in \mathbb{C}^{n \times n}$ *is invertible, then there exists*

- *a permutation matrix* P,
- *a lower-triangular matrix* L,
- *an upper-triangular matrix* U,

such that

$$PA = LU$$

More pivoting

Row pivoting is not the only option.

For example, *full* pivoting permutes *both* lower rows and rightmost columns in search of a maximum-magnitude pivot.

$$A = P_1 L_1 \; P_2 L_2 \; \cdots \; P_{n-1} L_{n-1} \, U \, Q_{n-1} Q_{n-2} \cdots Q_1,$$

where both P_j and Q_j are permutation matrices.

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An alternative is *rook pivoting*, which performs a permutation similar to the above, except that at elimination step j, the maximum is sought *only* over row j and column j.