The LU factorization

MATH 6610 Lecture 15

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Trefethen & Bau: Lecture 20

Gaussian elimination

Let $A \in \mathbb{C}^{n \times n}$ be an invertible matrix, and let $b \in \mathbb{C}^n$ be any vector.

Our goal is to compute the solution $x \in \mathbb{C}^n$ to the linear system,

$$Ax = b$$

Gaussian elimination

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One "standard" way to do this starts by forming the *augmented* rectangular matrix

$$(A \;\; b) \in \mathbb{C}^{n \times (n+1)},$$

and proceeds to perform elimination steps to transform the left $n \times n$ block into the identity matrix.

$$\left(A \;\vdots\; b \right) \xrightarrow[\text{operations}]{\text{row}} \left(I_{n \times n} \;\vdots\; c \right)$$

$$c = A^{-1} b \; (= x)$$

Row operations, I

If we record the row operations needed to perform Gaussian elimination, then we can work *only* on the matrix A.

Consider a matrix A with columns $(a_j)_{j=1}^n$:

$$A = \begin{pmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{pmatrix}, \qquad a_j = \begin{pmatrix} a_{j,1} \\ a_{j,2} \\ \vdots \\ a_{j,n} \end{pmatrix} \in \mathbb{C}^n$$

$a_{j,k} \rightarrow$ column j, row k

Gaussian elimination: first step: bring A to upper-triangular form.

$$A = \begin{pmatrix} a_{1,1} & a_{2,1} & \cdots & a_{n,1} \\ a_{1,2} & a_{2,2} & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,n} & a_{2,n} & \cdots & a_{n,n} \end{pmatrix} \xrightarrow[j=2,\dots,n]{\text{row}_j \leftarrow \text{row}_j - \frac{a_{1,j}}{a_{1,1}} (\text{row } 1)}$$

(assume $a_{1,1} \neq 0$)

$$\rightarrow \begin{pmatrix} a_{1,1} & a_{2,1} & \cdots & a_{n,1} \\ 0 & x & \cdots & x \\ 0 & x & & x \\ \vdots & \vdots & & \vdots \\ 0 & x & \cdots & x \end{pmatrix} = A_2$$

$\exists\ L_1 \in \mathbb{C}^{n \times n}$ s.t. $L_1 A_2 = A$

what is L_1?

Row j of L_1 has instructions about how to add together the rows of A_2 to recover row j of A.

$$L_1 = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \frac{a_{1,2}}{a_{1,1}} & 1 & 0 & \cdots & 0 \\ \frac{a_{1,3}}{a_{1,1}} & 0 & 1 & 0 \cdots & 0 \\ \vdots & & & & \\ \frac{a_{1,n}}{a_{1,1}} & 0 & \cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} | & | & | & & | \\ l_1 & e_2 & e_3 & \cdots & e_n \\ | & | & | & & | \end{pmatrix}$$

...

$$A = L_1 A_2 \qquad A_2 = \begin{pmatrix} a_{1,1} & | & & | \\ 0 & a_2^{(2)} & \cdots & a_n^{(2)} \\ \vdots & & & \\ 0 & | & & | \end{pmatrix}$$

Use $a_{2,2}^{(2)}$ to eliminate $a_{2,j}^{(2)}$ $j = 3 \dots n$
(if $a_{22}^{(2)} \neq 0$)

$$A_2 = L_2 A_3$$

result of elimination

$$A_3 = \begin{pmatrix} x & x & | & | & & | \\ 0 & x & a_3^{(3)} & a_4^{(3)} & \cdots & a_n^{(3)} \\ | & 0 & & & & \\ | & \vdots & & & & \\ 0 & 0 & | & | & & | \end{pmatrix}$$

$$A = L_1 A_2 = L_2 L_1 A_3 = \cdots = L_{n-1} L_{n-2} \cdots L_2 L_1 A_n$$

continue elimination

upper triangular

$$L_2 = \begin{pmatrix} | & | & | & & | \\ e_1 & \ell_2 & e_3 & \cdots & e_n \\ | & | & | & & | \end{pmatrix} \qquad \ell_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{a_{2,3}^{(2)}}{a_{2,2}^{(2)}} \\ \frac{a_{2,4}^{(2)}}{a_{2,2}^{(2)}} \\ \vdots \end{pmatrix}$$

Note: L_1, L_2 are lower triangular.

Row operations, II

After row operations that transform the first column to a multiple of e_1, we have

$$A = L_1 A_2, \qquad\qquad L_1 = \begin{pmatrix} | & - & 0_{1\times(n-1)} & - \\ \ell & & & \\ & & I_{(n-1)\times(n-1)} & \\ | & & & \end{pmatrix},$$

with A_2 the matrix

$$A_2 = \begin{pmatrix} a_{1,1} & | & & | \\ 0 & & & \\ & a_2^{(2)} & \cdots & a_n^{(2)} \\ \vdots & & & \\ 0 & | & & | \end{pmatrix}.$$

Row operations, III

If we continue triangular elimination from A_2, until the last column we obtain,

$$A = L_1 \cdots L_{n-1} A_n,$$

diary of row operations.

where A_n is an upper-triangular matrix, and each L_j has the form,

$$L_j = \begin{pmatrix} | & & | & | & | & & | \\ e_1 & \cdots & e_{j-1} & \ell_j & e_{j+1} & \cdots & e_n \\ | & & | & | & | & & | \end{pmatrix}$$

and $\ell_{j,j} = 1$.

Row operations, III

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$$L_j = \begin{pmatrix} | & & | & | & | & & | \\ e_1 & \cdots & e_{j-1} & \ell_j & e_{j+1} & \cdots & e_n \\ | & & | & | & | & & | \end{pmatrix}$$

Note that each L_j is lower triangular, and one can show that

$$L_j \overset{L_{j+1}}{\cancel{L_j +}} 1 = \begin{pmatrix} e_1 & \cdots & e_{j-1} & \ell_j & \ell_{j+1} & e_{j+2} & \cdots & e_n \end{pmatrix},$$

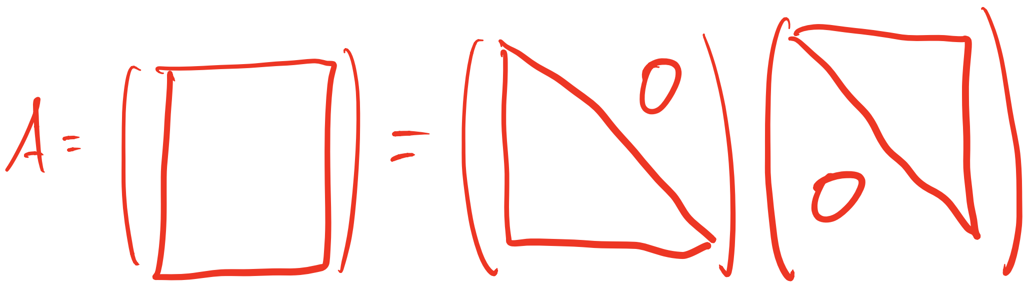
so that $L := \prod_{j=1}^{n-1} L_j$ is also lower- ~~upp~~er-triangular.

The LU factorization

We have just shown that, if all our elimination steps successfully complete, then

$$A = LU,$$

where L is lower-triangular, and U is upper-triangular.



The LU factorization

We have just shown that, if all our elimination steps successfully complete, then

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How can the steps fail? (I.e. $a_{j,j}^{(j)} \neq 0 \quad \forall\, j = 1 \ldots n-1$)

Theorem

A has an LU decomposition if and only if $\det A_j \neq 0$ for all $j = 1, \ldots, n$, where A_j is the principal (upper-left) $j \times j$ submatrix of A.

(A_j not the same as previous slides)

LU factorization utility

The LU factorization/decomposition has several uses;

- It's how we solve linear systems

1.) Compute $A = LU$

2.) Solve for y: $Uy = b$ (triangular solve)

3.) Solve for x: $Lx = y$ (triangular solve)

LU factorization utility

The LU factorization/decomposition has several uses;

- It's how we solve linear systems
- If an LU factorization for A is available, then solving $Ax = b$ requires only $\mathcal{O}(n^2)$ operations.

But: computing $A=LU$ requires $\mathcal{O}(n^3)$ operations.

LU factorization utility

The LU factorization/decomposition has several uses;

- It's how we solve linear systems
- If an LU factorization for A is available, then solving $Ax = b$ requires only $\mathcal{O}(n^2)$ operations.
- $\det A = \det L \det U$. (today: $\det L = 1$)

$O(n^3)$

Pencil-paper computation: Laplace expansion

$\Downarrow$ complexity

$O(n!)$