The LU factorization

MATH 6610 Lecture 15

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Trefethen & Bau: Lecture 20

Gaussian elimination

Let $A \in \mathbb{C}^{n \times n}$ be an invertible matrix, and let $b \in \mathbb{C}^n$ be any vector.

Our goal is to compute the solution $x \in \mathbb{C}^n$ to the linear system,

$$Ax = b$$

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One "standard" way to do this starts by forming the *augmented* rectangular matrix

$$(A \;\; b) \in \mathbb{C}^{n \times (n+1)},$$

and proceeds to perform elimination steps to transform the left $n \times n$ block into the identity matrix.

Row operations, I

If we record the row operations needed to perform Gaussian elimination, then we can work *only* on the matrix A.

Consider a matrix A with columns $(a_j)_{j=1}^n$:

$$A = \begin{pmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{pmatrix}, \qquad a_j = \begin{pmatrix} a_{j,1} \\ a_{j,2} \\ \vdots \\ a_{j,n} \end{pmatrix}$$

Row operations, II

After row operations that transform the first column to a multiple of e_1, we have

$$A = L_1 A_2, \qquad\qquad L_1 = \left(\begin{array}{cccc} | & - & 0_{1\times(n-1)} & - \\ \ell & & I_{(n-1)\times(n-1)} & \\ | & & & \end{array}\right),$$

with A_2 the matrix

$$A_2 = \left(\begin{array}{cccc} a_{1,1} & | & & | \\ 0 & & & \\ \vdots & a_2^{(2)} & \cdots & a_n^{(2)} \\ 0 & | & & | \end{array}\right).$$

Row operations, III

If we continue triangular elimination from A_2, until the last column we obtain,

$$A = L_1 \cdots L_{n-1} A_n,$$

where A_n is an upper-triangular matrix, and each L_j has the form,

$$L_j = \begin{pmatrix} | & & | & | & | & & | \\ e_1 & \cdots & e_{j-1} & \ell_j & e_{j+1} & \cdots & e_n \\ | & & | & | & | & & | \end{pmatrix}$$

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Note that each L_j is lower triangular, and one can show that

$$L_j L_j + 1 = \begin{pmatrix} e_1 & \cdots & e_{j-1} & \ell_j & \ell_{j+1} & e_{j+2} & \cdots & e_n \end{pmatrix},$$

so that $L := \prod_{j=1}^{n-1} L_j$ is also upper-triangular.

The LU factorization

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How can the steps fail?

Theorem

A has an LU decomposition if and only if $\det A_j \neq 0$ for all $j = 1, \ldots, n$, where A_j is the principal (upper-left) $j \times j$ submatrix of A.

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- If an LU factorization for A is available, then solving $Ax = b$ requires only $\mathcal{O}(n^2)$ operations.
- $\det A = \det L \det U$.