Least-squares problems

MATH 6610 Lecture 14

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Trefethen & Bau: Lecture 11

Least-squares problems

If $A \in C^{m \times n}$ and $b \in \mathbb{C}^n$, we are interested in computing the least-squares solution to

$$Ax = b$$

This arises in several situations, e.g., data fitting.

Least-squares solutions

The following is a result we have essentially already proven:

Theorem

Suppose $A \in \mathbb{C}^{m \times n}$ *has full column rank* (n)*. Then, for any* $b \in \mathbb{C}^n$*, there is a unique solution* x *that solves*

$$Ax = b$$

in the least-square sense. Furthermore, this solution x *is the unique solution to* $A^*Ax = A^*b$*, and the residual* $r := b - Ax$ *is orthogonal to* $\mathrm{range}(A)$*.*

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The system $A^* A x = A^* b$ is called the *normal equations*.

Computational solutions

While the normal equations are typically useful for analysis, they are typically not used for computation.

$$A = QR \qquad \Longrightarrow \qquad x = R^{-1}Q^{*}b.$$

In most cases, the QR decomposition is used, largely for stability reasons.