The QR decomposition

MATH 6610 Lecture 11

September 28, 2020

Trefethen & Bau: Lecture 7

Orthogonalization

The main goal of orthogonalization:

Given $\{a_j\}_{j=1}^n \subset \mathbb{C}^m$, compute $\{q_j\}_{j=1}^n$ such that:

$$\langle q_j, q_k \rangle = \delta_{j,k}, \qquad \text{span}\{a_1, \ldots, a_n\} = \text{span}\{q_1, \ldots, q_n\}$$

Orthogonalization

The main goal of orthogonalization:

Given $\{a_j\}_{j=1}^n \subset \mathbb{C}^m$, compute $\{q_j\}_{j=1}^n$ such that:

$$\langle q_j, q_k \rangle = \delta_{j,k}, \qquad \mathrm{span}\{a_1, \ldots, a_n\} = \mathrm{span}\{q_1, \ldots, q_n\}$$

Why?

One reason is that the $\mathbb{C}^m$-orthogonal projector onto $\mathrm{span}\{a_1, \ldots, a_n\}$ is given by,

$$P = QQ^*, \qquad Q = \begin{pmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{pmatrix}$$

Gram-Schmidt orthogonalization

One essentially explicit algorithm to orthogonalize is Gram-Schmidt.

Input: n vectors $\{a_j\}_{j=1}^n$. (Assume they're linearly independent for now.)
Output: n vectors $\{q_j\}_{j=1}^n$.

Gram-Schmidt to QR

$$\{a_1, \ldots, a_n\} \quad \longrightarrow \quad \{q_1, \ldots, q_n\},$$
$$\operatorname{span}\{a_1, \ldots, a_j\} = \operatorname{span}\{q_1, \ldots, q_j\} \ \ (1 \leqslant j \leqslant n)$$

Gram-Schmidt to QR

$$\{a_1, \ldots, a_n\} \quad \longrightarrow \quad \{q_1, \ldots, q_n\},$$
$$\operatorname{span}\{a_1, \ldots, a_j\} = \operatorname{span}\{q_1, \ldots, q_j\} \;\; (1 \leqslant j \leqslant n)$$

We can rewrite this to explicitly express the original vectors a_j in terms of the orthogonalized vectors q_j.

The QR decomposition

In fact, these computations implies the following result:

Theorem

Let $A \in \mathbb{C}^{m\times n}$ *be any matrix. Then there exists a unitary matrix* $Q \in \mathbb{C}^{m\times m}$*, and an upper-triangular matrix* $R \in \mathbb{C}^{m\times n}$ *such that*

$$A = QR.$$

If A *has full rank, then the diagonal entries of* R *can be chosen to be positive.*