The singular value decomposition

MATH 6610 Lecture 09

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Trefethen & Bau: Lectures 4, 5

Diagonalizability

Recall:

- All non-defective square matrices are diagonalizable (eigenvalue decomposition)
- All square matrices are bidiagonalizable (Jordan normal form)
- All square matrices are unitarily triangularizable (Schur decomposition)
- All normal matrices are unitarily diagonalizable (spectral theorem)

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What about rectangular matrices?

The singular value decomposition

Theorem (SVD)

Any matrix $A \in \mathbb{C}^{m\times n}$ can be written as the product,

$$A = U\Sigma V^*,$$

where $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary.
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With $p = \min\{m, n\}$, notational convention:

- $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_p)$
- $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_p \geqslant 0$
- $U = [u_1,\ u_2,\ \ldots,\ u_m]$
- $V = [v_1,\ v_2,\ \ldots,\ v_n]$

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If A is square and invertible:

$$\kappa(A) = \frac{\sigma_1}{\sigma_n}$$