The spectral theorem

MATH 6610 Lecture 08

September 21, 2020

Eigenvalues/eigenvectors

Trefethen & Bau: Lecture 24

Diagonalizability

Recall:

- All non-defective square matrices are diagonalizable (eigenvalue decomposition) ✓
- All square matrices are bidiagonalizable (Jordan normal form)
- All square matrices are unitarily triangularizable (Schur decomposition)

nonzeros on main, super-diagonal.

A triangular: $A = \begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix}$ (upper triangular)

EV decomposition: $SAS^{-1} = \begin{pmatrix} \ddots & 0 \\ 0 & \ddots \end{pmatrix}$

Jordan normal form: $SAS^{-1} = \begin{pmatrix} \ddots & 0 \\ 0 & \ddots \end{pmatrix}$

Schur decomp: $UAU^* = \begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix}$

Diagonalizability

Recall:

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When are matrices unitarily diagonalizable?

$$UAU^* = \begin{pmatrix} \ddots & 0 \\ 0 & \ddots \end{pmatrix}$$

The spectral theorem

Definition

A matrix $A \in \mathbb{C}^{n\times n}$ is a *normal matrix* if it commutes with its transpose.

$$AA^* = A^*A$$

1.) Hermitian & skew-Hermitian matrices are normal

2.) Unitary matrices are normal $(UU^* = I = U^*U)$

3.) There are normal matrices that don't fit either of the above classifications

$$\begin{pmatrix} 1+i & 0 \\ 0 & -1-i \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

The spectral theorem

Definition

A matrix $A \in \mathbb{C}^{n\times n}$ is a *normal matrix* if it commutes with its transpose.

"for normal matrices"

Theorem (The spectral theorem)

A square matrix A is unitarily diagonalizable if and only if it is normal.

Proof: Assume A is unitarily diagonalizable:

$A = U\Lambda U^*$ $A^* = U\Lambda^* U^*$

$AA^* = U\Lambda\Lambda^* U^*$

$A^*A = U\Lambda^*\Lambda U^*$

Also: $\Lambda\Lambda^* = \Lambda^*\Lambda$ (direct computation)

$\Rightarrow A^*A = AA^*$
(A is normal)

Assume A is normal. 2 steps

(i) Reduce problem to analyzing normal upper triangular matrices

(ii) Show a normal, upper triangular matrix is diagonal.

(i) Schur decomposition: (HW)

$$A = UTU^*, \quad U \text{ unitary}$$
$$T: \text{(upper) triangular}$$

A is normal: $UTT^*U^* = AA^* = A^*A = UT^*TU^*$

$\Longrightarrow TT^* = T^*T$ (T is normal) ✓

(ii) T is normal and upper triangular.

$$TT^* = T^*T \Longrightarrow (TT^*)_{i,i} = (T^*T)_{i,i}$$
$$i = 1 \ldots n.$$

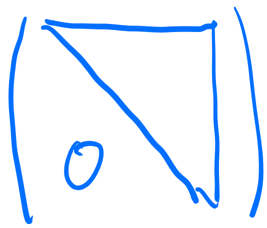
$$(TT^*)_{i,i} = (T^*T)_{i,i}$$

$$e_i^* TT^* e_i \qquad e_i^* T^* T e_i$$

$$\langle T^* e_i, T^* e_i \rangle \qquad \langle Te_i, Te_i \rangle$$

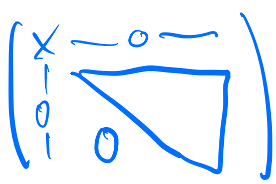
(e_i = vector of zeros, with entry 1 in ith place)

$$\underbrace{\|T^* e_i\|_2^2}_{\text{norm of ith row of } T} = \underbrace{\|T e_i\|_2^2}_{\text{norm of ith column of } T.}$$

$T =$ 

$i=1$: column 1 and row 1 of T have same norm.

$\Rightarrow$ entries $2, \ldots, n$ in row 1 must be 0,

$T =$ 

$i=2$: column 2 and row 2 have same norm

$\Rightarrow$ entries $3, \ldots, n$ in row 2 must be 0.

$\vdots$ continue (finite induction).

$\Rightarrow T$ is diagonal.

$A = UTU^*$, T is diagonal and U is unitary.

$\blacksquare$

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- then the 2-norm induced operator, spectral radius, and maximum numerical range coincide

$$\|A\|_2 = \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} \qquad \rho(A) = \max_j |\lambda_j|$$

$$W_A(\mathbb{C}^n) = \left\{ \frac{\langle Ax, x \rangle}{\langle x, x \rangle} \;\middle|\; x \in \mathbb{C}^n \setminus \{0\} \right\}$$

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- then $A = A^*U$ for some unitary U
- $\|A\|_F^2 = \sum_{i,j=1}^{n} |a_{i,j}|^2 = \sum_{j=1}^{n} |\lambda_j|^2$

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Actually, any of the above also implies that A is normal.

Eigenvalue conditioning

eigenvalue-computing map

Consider $A \in \mathbb{C}^{n\times n}$, along with the map $\lambda : \mathbb{C}^{n\times n} \to \mathbb{C}$ that computes an eigenvalue of A:

$$Av = \lambda(A)v, \qquad \text{for some } v \neq 0$$

What is the conditioning of this operation?

How sensitive is $A \mapsto \lambda(A)$ to perturbations in A?

Eigenvalue conditioning

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Theorem (Bauer-Fike)

Assume $A \in C^{n\times n}$ is diagonalizable with eigenvector matrix $V \in \mathbb{C}^{n\times n}$. Then, using the 2-norm, the absolute condition number of computing eigenvalues is $\widetilde{\kappa}_\lambda(A) = \kappa(V)$.

HW

$$V = \underbrace{\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}}_{\text{eigenvectors}}$$

$$\kappa(V) = \|V\|_2 \|V^{-1}\|_2$$

$$\widetilde{\kappa}_\lambda(A) := \lim_{\delta \to 0} \sup_{\|\delta A\|_2 \leq \delta} \frac{\|\lambda(A+\delta A) - \lambda(A)\|_2}{\|\delta A\|_2}$$

$$\| \quad \kappa(V)$$

If A is normal: $K(V) = 1$ $\quad (\|V\|_2 = \|V^{-1}\|_2 = 1)$

$\Rightarrow \tilde{K}_\lambda(A) = 1$