The spectral theorem

MATH 6610 Lecture 08

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Trefethen & Bau: Lecture 24

Diagonalizability

Recall:

- All non-defective square matrices are diagonalizable (eigenvalue decomposition)
- All square matrices are bidiagonalizable (Jordan normal form)
- All square matrices are unitarily triangularizable (Schur decomposition)

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When are matrices unitarily diagonalizable?

The spectral theorem

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Theorem (The spectral theorem)

A square matrix A is unitarily diagonalizable if and only if it is normal.

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Actually, any of the above also implies that A is normal.

Eigenvalue conditioning

Consider $A \in \mathbb{C}^{n \times n}$, along with the map $\lambda : \mathbb{C}^{n \times n} \to \mathbb{C}$ that computes an eigenvalue of A:

$$Av = \lambda(A)v, \qquad \text{for some } v \neq 0$$

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Theorem (Bauer-Fike)

Assume $A \in C^{n \times n}$ is diagonalizable with eigenvector matrix $V \in \mathbb{C}^{n \times n}$. Then, using the 2-norm, the absolute condition number of computing eigenvalues is $\widetilde{\kappa}_\lambda(A) = \kappa(V)$.