Numerical stability

MATH 6610 Lecture 07

September 18, 2020

Trefethen & Bau: Lecture 14, 15

Numerical algorithms

Given $f : \mathbb{C}^n \to \mathbb{C}^m$, we wish to understand how roundoff errors affect evaluation of f.

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- Loss of accuracy due to finite precision
- Conditioning of f

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Let $\widetilde{f} : \mathbb{C}^n \to \mathbb{C}^m$ denote the actual algorithmic implementation of f.

We might hope that

$$\frac{\|f(x) - \widetilde{f}(x)\|}{\|f(x)\|} = \mathcal{O}(\epsilon_{\text{mach}}),$$

but this is typically too much to ask for if f is ill-conditioned.

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and this motivates a definition.

Definition

An algorithm $\widetilde{f}$ is forward stable if, for every $x \in \mathbb{C}^n$, we have

$$\frac{\|\widetilde{f}(x) - f(\widetilde{x})\|}{\|f(\widetilde{x})\|} = \mathcal{O}(\epsilon_{\mathrm{mach}}),$$

for some $\widetilde{x}$ satisfying $\|x - \widetilde{x}\| = \|x\|\mathcal{O}(\epsilon_{\mathrm{mach}})$.

A forward stable algorithm gives an "approximately correct answer to a closely related question."

Forward stability, II

$$\frac{\|\widetilde{f}(x) - f(\widetilde{x})\|}{\|f(\widetilde{x})\|} = \mathcal{O}(\epsilon_{\text{mach}}), \tag{1}$$

for some $\widetilde{x}$ satisfying $\|x - \widetilde{x}\| = \|x\|\mathcal{O}(\epsilon_{\text{mach}})$.

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If $\widetilde{f}$ is forward stable, we can show that $\widetilde{f}$ produces a reasonable approximation to f.

This error estimation procedure, requiring the establishment of (1), is *forward error analysis*.

Showing forward stability (1) is frequently technical and difficult.

Backward stability

A dual, but somewhat stranger+stronger notion of stability is backward stability.

Definition

An algorithm $\widetilde{f}$ is backward stable if, for every $x \in \mathbb{C}^n$, we have

$$\widetilde{f}(x) = f(\widetilde{x}), \tag{2}$$

for some $\widetilde{x}$ satisfying $\|x - \widetilde{x}\| = \|x\| \mathcal{O}(\epsilon_{\text{mach}})$.

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A backward stable algorithm gives an "exact answer to a closely related question."

If $\widetilde{f}$ is backward stable, then showing accuracy of the algorithm $\widetilde{f}$ is easier.

Estimating error by establishing (2) is *backward error analysis*.

Backward stability is frequently easier to show than forward stability.

Examples

Assume some floating-point axioms:

1. For each $x \in \mathbb{R}$, then $fl(x) = x(1+\epsilon)$ for $|\epsilon| \leqslant \mathcal{O}(\epsilon_{\text{mach}})$.
2. For floating-points numbers $x, y \in \mathbb{R}$, then $x \odot y = (x \cdot y)(1+\epsilon)$ for $|\epsilon| \leqslant \mathcal{O}(\epsilon_{\text{mach}})$. $(\cdot = +,\ -,\ \times)$

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Example

The floating-point implementation of $f(x,y) = xy^T$ is *not* backward stable.