Conditioning of problems

MATH 6610 Lecture 06

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Trefethen & Bau: Lecture 12

Solution sensitivity

End goal (later): understand sensitivity of numerical algorithms to roundoff errors. ("stability")

due to floating-point precision.

First task (today): understand sensitivity of solutions of *mathematical* problems. ("conditioning")

a property only of the mathematical problem, not of finite precision.

Solution sensitivity

End goal (later): understand sensitivity of numerical algorithms to roundoff errors. ("stability")

First task (today): understand sensitivity of solutions of *mathematical* problems. ("conditioning")

Example

Let $f(x) = ax$ for scalars a, x. , $f : \mathbb{C} \to \mathbb{C}$

The sensitivity of the map $x \mapsto f(x)$ depends on the value of a.

$$f(x+\varepsilon) = ax + a\varepsilon$$
$$= f(x) + \boxed{a\varepsilon}$$

perturbation in the output

perturbation in input

Vector norms

We'll measure perturbations of functions (generally over $\mathbb{C}^n$), and will measure these perturbations with norms.

Theorem (Equivalence of norms)

Let $\|\cdot\|_a$ and $\|\cdot\|_b$ be norms on a finite dimensional vector space V. Then $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent norms.

In $\mathbb{C}^n$: $\|\cdot\|_2$ is fine

we also have $\|\cdot\|_p$, $(p \geq 1)$

Q: Does the definition of norm affect notions of conditioning?

A: (No, effectively) ☆

Norms $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent if $\exists\ 0 < c < C < \infty$

such that $c\|x\|_a \leq \|x\|_b \leq C\|x\|_a \quad \forall x \in \mathbb{C}^n$.

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For this reason, we'll consider a(n unspecified) generic norm $\|\cdot\|$ in what follows.

Absolute sensitivity measures

Let $f : \mathbb{C}^n \to \mathbb{C}^m$.

A sensible measure of sensitivity of f at x is perturbation-based:

$$\frac{\|f(x+\delta x) - f(x)\|}{\|\delta x\|}. \qquad \delta x \in \mathbb{C}^n$$

perturbation of output

perturbation of input.

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all directions in $\mathbb{C}^n$

As with derivatives, we can measure the sensitivity by taking limits:

$$\hat{\kappa}(x) := \lim_{\delta \to 0} \sup_{\|\delta x\| \leqslant \delta} \frac{\|\delta f\|}{\|\delta x\|}, \qquad \delta f := \frac{\cancel{\|}f(x+\delta x) - f(x)\cancel{\|}}{\cancel{\|\delta x\|}}.$$

$\hat{\kappa}(x)$ is called the <u>absolute condition number</u> of f at x.

$$\hat{\kappa}(x) = \hat{\kappa}_f(x)$$

$$\delta f = f(x+\delta x) - f(x)$$

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$\hat{\kappa}(x)$ is called the absolute condition number of f at x.

Note that condition numbers are properties of the map f and *not* of an algorithmic or finite-precision implementation.

Relative sensitivity measures

Recall: floating-point arithmetic makes relative errors, not absolute ones.

Thus, absolute condition numbers have limited utility for understanding numerical stability.

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Recall: floating-point arithmetic makes relative errors, not absolute ones.

Thus, absolute condition numbers have limited utility for understanding numerical stability.

sometimes just "condition number"

The relative condition number of f at x is defined as

relative error in input. (machine precision)

$\delta f = f(x+\delta x) - f(x)$

relative error in output

$$\kappa(x) := \lim_{\delta \to 0} \sup_{\|\delta x\| \leqslant \delta} \frac{\frac{\|\delta f\|}{\|f(x)\|}}{\frac{\|\delta x\|}{\|x\|}}$$

$$= \lim_{\delta \to 0} \sup_{\|\delta x\| \leqslant \delta} \frac{\|\delta f\| \, \|x\|}{\|\delta x\| \, \|f(x)\|}$$

Again, this is a (mathematical) property of f, and not of an algorithmic implementation.

Relative sensitivity measures

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Thus, absolute condition numbers have limited utility for understanding numerical stability.

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Problems (functions f) with "small" condition numbers are *well-conditioned*.
Problems (functions f) with "large" condition numbers are *ill-conditioned*.

Examples

Example

If $f : \mathbb{C}^n \to \mathbb{C}^m$ is smooth, then condition numbers are norms of Jacobians.

$$\hat{\kappa}(x) = \|J(x)\|, \qquad J(x) := \frac{\partial f}{\partial x}(x).$$

absolute cond. number.

matrix norm induced by norms on $\mathbb{C}^n$ and $\mathbb{C}^m$.

$J(x) \in \mathbb{C}^{m \times n}$

$(J(x))_{k,\ell} = \frac{\partial f_k}{\partial x_\ell}(x)$

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Example

$f : \mathbb{C} \to \mathbb{C}$ defined by $f(x) = ax$.

f is smooth: $\hat{\kappa}(x) = |f'(x)| = |a|$

$$\kappa(x) = \lim_{\delta \to 0} \sup_{\|\delta x\| \le \delta} \frac{\|\delta f\| / \|f\|}{\|\delta x\| / \|x\|}$$

don't depend on δx, δ.

$$= \hat{\kappa}(x) \cdot \frac{\|x\|}{\|f(x)\|} = |a| \frac{|x|}{|ax|} = 1.$$

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Example

$f : \mathbb{C} \to \mathbb{C}$ defined by $f(x) = ax$.

Example

$f : \cancel{\mathbb{C}} \to \mathbb{C}$ defined by $f(x) = x^p$ for arbitrary $p > 0$. (and $x > 0$)

$(0, \infty)$

f is smooth $\Rightarrow$ $\hat{K}(x) = |f'(x)| = p\,|x|^{p-1}$

$$K(x) = \hat{K}(x) \cdot \frac{|x|}{|f(x)|} = p\,|x|^{p-1} \cdot \frac{|x|}{|x|^p} = p.$$

Examples

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Example

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Example

$f : \mathbb{C}^n \to \mathbb{C}^n$ defined by $f(x) = Ax$ for invertible $A \in \mathbb{C}^{n \times n}$.

Use 2-norm on $\mathbb{C}^n$.

$$\hat{\kappa}(x) = \|J\| = \|A\|_2$$

$$\kappa(x) = \hat{\kappa}(x) \cdot \frac{\|x\|}{\|Ax\|} = \|A\| \frac{\|x\|}{\|Ax\|} \quad (\neq 1)$$

what is the most sensitive this map can be over all x?

$$\sup_{x \neq 0} \kappa(x) = \|A\| \cdot \sup_{x \neq 0} \frac{\|x\|}{\|Ax\|} \underset{y = Ax}{=} \|A\| \underbrace{\sup_{y \neq 0} \frac{\|A^{-1}y\|}{\|y\|}} = \|A\|_2 \|A^{-1}\|_2$$

Linear problems

The (relative) condition number of the linear map $x \mapsto Ax$, for invertible A, is bounded by

$$\sup_{x \in \mathbb{C}^n \setminus \{0\}} \kappa(x) = \|A\| \|A^{-1}\|.$$

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By a similar argument, given A and b, the condition number of the problem that finds the solution x to

$$Ax = b,$$ (consider map $b \mapsto A^{-1}b$)

is (bounded by) $\|A\| \|A^{-1}\|$.

Linear problems

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$$\sup_{x \in \mathbb{C}^n} \kappa(x) = \|A\| \|A^{-1}\|.$$

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$$Ax = b,$$

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More generally, given invertible A, the (matrix) condition number of A is defined as

$$\boxed{\kappa(A) = \|A\| \|A^{-1}\|.}$$