Conditioning of problems

MATH 6610 Lecture 06

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Trefethen & Bau: Lecture 12

Solution sensitivity

End goal (later): understand sensitivity of numerical algorithms to roundoff errors. ("stability")

First task (today): understand sensitivity of solutions of *mathematical* problems. ("conditioning")

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Example

Let $f(x) = ax$ for scalars a, x.
The sensitivity of the map $x \mapsto f(x)$ depends on the value of a.

Vector norms

We'll measure perturbations of functions (generally over $\mathbb{C}^n$), and will measure these perturbations with norms.

Theorem (Equivalence of norms)

Let $\|\cdot\|_a$ *and* $\|\cdot\|_b$ *be norms on a finite dimensional vector space* V*. Then* $\|\cdot\|_a$ *and* $\|\cdot\|_b$ *are equivalent norms.*

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For this reason, we'll consider a(n unspecified) generic norm $\|\cdot\|$ in what follows.

Absolute sensitivity measures

Let $f : \mathbb{C}^n \to \mathbb{C}^m$.

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$$\frac{\|f(x + \delta x) - f(x)\|}{\|\delta x\|}.$$

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As with derivatives, we can measure the sensitivity by taking limits:

$$\hat{\kappa}(x) := \lim_{\delta \to 0} \sup_{\|\delta x\| \leqslant \delta} \frac{\|\delta f\|}{\|\delta x\|}, \qquad \delta f := \frac{\|f(x + \delta x) - f(x)\|}{\|\delta x\|}.$$

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Note that condition numbers are properties of the map f and *not* of an algorithmic or finite-precision implementation.

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The relative condition number of f at x is defined as

$$\begin{aligned}\kappa(x) &\coloneqq \lim_{\delta\to 0} \sup_{\|\delta x\| \leqslant \delta} \frac{\frac{\|\delta f\|}{\|f(x)\|}}{\frac{\|\delta x\|}{\|x\|}} \\ &= \lim_{\delta\to 0} \sup_{\|\delta x\| \leqslant \delta} \frac{\|\delta f\| \, \|x\|}{\|\delta x\| \, \|f(x)\|}\end{aligned}$$

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Problems (functions f) with "small" condition numbers are *well-conditioned*.
Problems (functions f) with "large" condition numbers are *ill-conditioned*.

Examples

Example

If $f : \mathbb{C}^n \to \mathbb{C}^m$ is smooth, then condition numbers are norms of Jacobians.

$$\hat{\kappa}(x) = \|J(x)\|, \qquad J(x) := \frac{\partial f}{\partial x}(x).$$

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$f : \mathbb{C}^n \to \mathbb{C}^n$ defined by $f(x) = Ax$ for invertible $A \in \mathbb{C}^{n \times n}$.

Linear problems

The (relative) condition number of the linear map $x \mapsto Ax$, for invertible A, is bounded by

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More generally, given invertible A, the (matrix) condition number of A is defined as

$$\kappa(A) = \|A\|\|A^{-1}\|.$$