(IEEE) Floating-point representation

MATH 6610 Lecture 05

September 11, 2020

(Lecture 13 in Trefethen & Bau)

Finite representation of numbers

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Without context, the base b can be other positive integers:

$$b = 6 \quad \Rightarrow \quad 3 \times 6^1 \;+\; 4 \times 6^0 \;+\; 1 \times 6^{-1} \;+\; 5 \times 6^{-2} \;+\; 0 \times 6^{-3} \;+\; 3 \times 6^{-4}$$

Computer representations

Circuits typically detect the presence (1) or absence (0) of an electrical signal.

For this reason, base $b = 2$ is the standard computer format, e.g.,:

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(b=2 ⇒ "binary")

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8 bits = 1 "byte" (Above: 19 bits)

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Binary representations must store the integer part, the fractional part, and typically also a sign.

Fixed point vs floating point

Fixed-point representations have a fixed radix point, with the size of the fractional part predetermined:

$$100010.\quad \underbrace{0010011001111}_{\text{fixed number of entries}}$$

Then the fractional precision of this representation is fixed for any number.

This *truncation* of finite representations is one of the main challenges to address in numerical computations.

Above: 6 bits for integer part
13 bits for fractional part.

Fixed precision: can't represent numbers smaller than 2^{-13}
(or larger than 2^{7})

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Floating-point representations allow the radix to float:

"radix shift"

$$1.\underbrace{0001000100110011111}_{\text{fixed number of entries}} \quad + \quad \text{exponent} \quad 5$$

e.g. exponent $\cancel{6}$ $\Rightarrow$

The exponent encodes which exponent the radix is aligned with. (Above: 5)

Floating-point representations

encodes integer + fractional part

Generally speaking, floating-point representations store:

$$\color{blue}{\text{significand}} + \color{red}{\text{exponent}} + \text{sign}$$

The significand combines the integer and fractional parts of the number.

The exponent encodes the location of the radix.

Floating-point representations allows for a (much) larger operating range than fixed-point representations.

exponent

In decimal: $\pm I \cdot 10^{E}$

significand

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The most popular representation is the IEEE 754 standard, defining various formats:

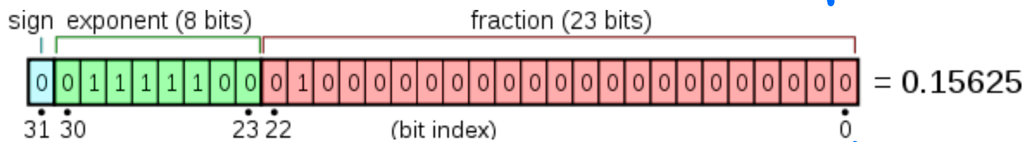
- Binary 16 # of bits used for storage.
- Binary 32 ("single precision")
- Binary 64 ("double precision") ☆ (most of the time)
- ⋮

Most high-level scientific computing languages use double precision as default.

Floating-point details

32 bits (single precision)

These standards define bit allocation:



Source: `https://en.wikipedia.org/wiki/File:Float_example.svg`

significand

exponent: encoded as a signed integer.

significand: encoded as an unsigned integer

DP: exponent: 11 bits

significand: 52 bits

In DP: what's the largest # I can represent?

$I \cdot 2^E$, Exponent E: 11 bits

Encodes integers between -2^{10} and 2^{10}.

Ignoring significand: 2^E can be as large as $2^{2^{10}} = 2^{1024}$

Similarly: smallest number (in magnitude) is $2^{-2^{10}} = 2^{-1024}$.

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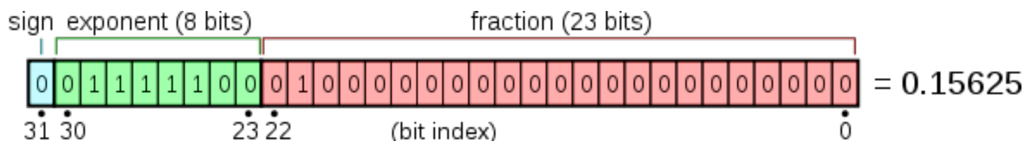

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Machine precision or machine epsilon is the maximum relative rounding error due to finite representation,

$$\left| \frac{x - \mathrm{fl}(x)}{x} \right| .$$

floating-point representation of x.

Roughly speaking, machine epsilon is also the largest ϵ such that $1 + \epsilon$ is rounded to 1.

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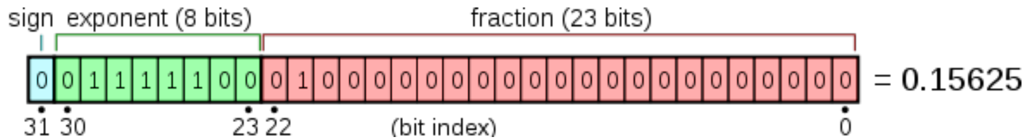

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base^{-E}, $-E$: smallest integer representable by significand.

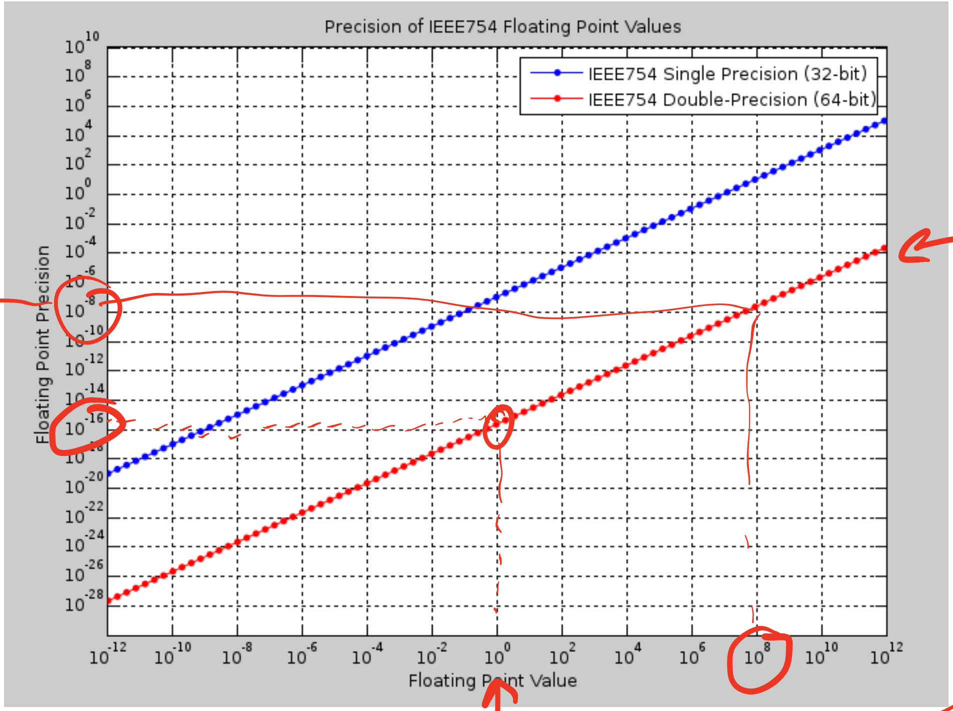
The roundoff or truncation error associated with a floating-point format is essentially equal to $\text{base}^{-\#(\text{exponent digits})}$.

Machine precision

Floating-point representation ensures that there is not really an absolute error committed by computer representations, there is only a relative error.

$\approx 10^{-8}$

$\approx 10^{-16}$



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Roundoff and computing

The rounding truncation imparted by finite representations requires attention to how algorithms are implemented. E.g.,

- Implementation of the formula $\underbrace{\sqrt{1+x^4}-1}$ for small positive x.
- Evaluation of e^x for $x < 0$.
- Evaluation of $\frac{f(x+h)-f(x)}{h}$ for small h.

$f(x) := \sqrt{1+x}$

Taylor Series around $x=0$

$f(x) \approx 1 + \frac{x}{2}$

$\Rightarrow \sqrt{1+x^4} \approx 1+\frac{x^4}{2}$ for x^4 close to 0.

$\underbrace{\sqrt{1+x^4}}-1 \approx \frac{x^4}{2}$ for x^4 close to 0.