(IEEE) Floating-point representation

MATH 6610 Lecture 05

September 11, 2020

Finite representation of numbers

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Without context, the base b can be other positive integers:

$$b = 6 \quad \Rightarrow \quad 3 \times 6^{1} \; + \; 4 \times 6^{0} \; + \; 1 \times 6^{-1} \; + \; 5 \times 6^{-2} \; + \; 0 \times 6^{-3} \; + \; 3 \times 6^{-4}$$

Computer representations

Circuits typically detect the presence (1) or absence (0) of an electrical signal.

For this reason, base $b = 2$ is the standard computer format, e.g.,:

$$100010.0010011001111$$

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Binary representations must store the integer part, the fractional part, and typically also a sign.

Fixed point vs floating point

Fixed-point representations have a fixed radix point, with the size of the fractional part predetermined:

$$100010.\quad \underbrace{0010011001111}_{\text{fixed number of entries}}$$

Then the fractional precision of this representation is fixed for any number.

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Floating-point representations allow the radix to float:

$$1.\underbrace{0001000100110011111}_{\text{fixed number of entries}} \quad + \quad \text{exponent}$$

The exponent encodes which exponent the radix is aligned with. (Above: 5)

Floating-point representations

Generally speaking, floating-point representations store:

$$\text{significand} + \text{exponent} + \text{sign}$$

The significand combines the integer and fractional parts of the number.

The exponent encodes the location of the radix.

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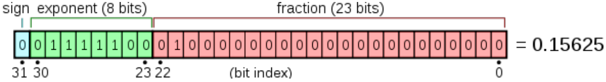
The most popular representation is the IEEE 754 standard, defining various formats:

- Binary 16
- Binary 32 ("single precision")
- Binary 64 ("double precision")
- $\vdots$

Most high-level scientific computing languages use double precision as default.

Floating-point details

These standards define bit allocation:



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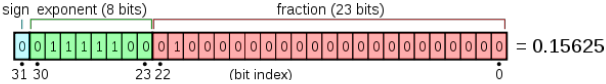

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Machine precision or machine epsilon is the maximum relative rounding error due to finite representation,

$$\left| \frac{x - \mathrm{fl}(x)}{x} \right|.$$

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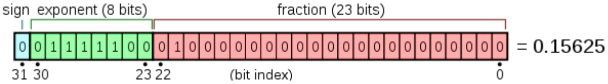

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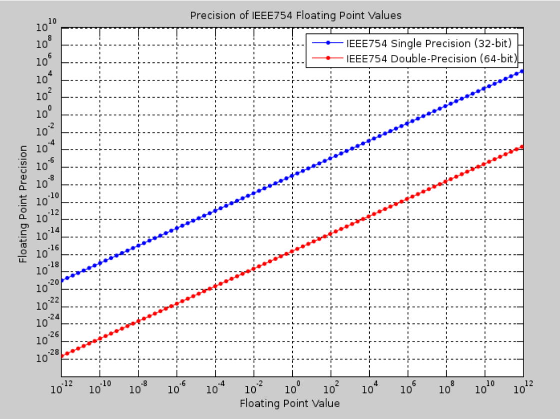
The roundoff or truncation error associated with a floating-point format is essentially equal to $\mathrm{base}^{-\#(\text{exponent digits})}$.

Machine precision

Floating-point representation ensures that there is not really an absolute error committed by computer representations, there is only a relative error.



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Roundoff and computing

The rounding truncation imparted by finite representations requires attention to how algorithms are implemented. E.g.,

- Implementation of the formula $\sqrt{1+x^4}-1$ for small positive x.
- Evaluation of e^x for $x < 0$.
- Evaluation of $\frac{f(x+h)-f(x)}{h}$ for small h.