Variational characterizations of eigenvalues

MATH 6610 Lecture 04

September 9, 2020

Hermitian matrices

If $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ is Hermitian, recall that:

- $\boldsymbol{A}$ is unitarily diagonalizable, $\boldsymbol{A} = \boldsymbol{U} \boldsymbol{\Lambda} \boldsymbol{U}^*$
- The spectrum of $\boldsymbol{A}$ is real-valued
- (Hermitian) Positive-definite matrices have matrix square roots

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Today: Variational characterizations of eigenvalues for Hermitian matrices.

Rayleigh quotients, I

Let $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ be a matrix, and let $\boldsymbol{x} \in \mathbb{C}^n \backslash \{\boldsymbol{0}\}$ be a vector.

The Rayleigh Quotient (of $\boldsymbol{A}$ at $\boldsymbol{x}$) is the scalar,

$$R_{\boldsymbol{A}}(\boldsymbol{x}) := \frac{\langle \boldsymbol{A}\boldsymbol{x}, \boldsymbol{x} \rangle}{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}$$

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We can also consider the image of the Rayleigh quotient but only on a subspace V:

$$W_{\boldsymbol{A}}(V) := R_{\boldsymbol{A}}\left(V \backslash \{\mathbf{0}\}\right).$$

Rayleigh quotients, II

Let $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ be Hermitian. Consider a subspace $V \subset \mathbb{C}^n$.

The image of the V under the Rayleigh quotient, $W_{\boldsymbol{A}}(V)$, is some subset of $\mathbb{R}$.

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- The maximum of $W_{\boldsymbol{A}}(V)$ can be $\lambda_{\max}(\boldsymbol{A})$.
  What is the smallest possible maximum value?

The "min-max" theorem

Theorem (Courant-Fischer-Weyl "min-max")

Let $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ be Hermitian, with eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$. Then for each $1 \leqslant k \leqslant n$,

$$\lambda_k = \min_{\substack{V \subset \mathbb{C}^n \\ \dim V = k}} \max W_{\boldsymbol{A}}(V)$$

$$\lambda_k = \max_{\substack{V \subset \mathbb{C}^n \\ \dim V = n-k+1}} \min W_{\boldsymbol{A}}(V)$$

In addition, if $(\boldsymbol{u}_j)_{j=1}^n$ are the eigenvectors associated with $(\lambda_j)_{j=1}^n$, then:

- $V = \mathrm{span}\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k\}$ *achieves the outer minimum*
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Cauchy interlacing theorem

A matrix $\boldsymbol{B}$ is a compression of $\boldsymbol{A}$ if $\boldsymbol{B} = \boldsymbol{Q}^* \boldsymbol{A}\boldsymbol{Q}$ for some $\boldsymbol{Q} \in \mathbb{C}^{n \times r}$ with othonormal columns.

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Just one consequence of the min-max theorem:

Theorem (Cauchy interlacing)

Let $\boldsymbol{B} \in \mathbb{C}^{(n-1)\times(n-1)}$ be a compression of a Hermitian matrix $\boldsymbol{A} \in \mathbb{C}^{n \times n}$. If $\boldsymbol{A}$ has eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$, and $\boldsymbol{B}$ has eigenvalues $\mu_1, \ldots, \mu_{n-1}$, then

$$\lambda_j \leqslant \mu_j \leqslant \lambda_{j+1},$$

for all $j = 1, \ldots, n-1$.