Eigendecompositions of Hermitian matrices

MATH 6610 Lecture 03

September 4, 2020

Diagonalizability

Recall:

- All non-defective square matrices are diagonalizable (eigenvalue decomposition)
- All square matrices are bidiagonalizable (Jordan normal form)
- All square matrices are unitarily triangularizable (Schur decomposition)

"most" matrices are diagonalizable.

Diagonalizability

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When are matrices unitarily diagonalizable?

A spectral theorem

Theorem ("Spectral thm for Hermitian matrices")
If $A \in \mathbb{C}^{n\times n}$ is Hermitian, then it is unitarily diagonalizable with real eigenvalues.

Hermitian matrices are also called *self-adjoint*. $(A = A^*)$

I.e., $A = A^* \Longrightarrow \exists$ unitary matrix U s.t.

$A = U \Lambda U^*$, Λ is diagonal with real entries.

Proof: first: show λ (an eigenvalue) is real.

Let (λ, v) be an eigenpair $(\|v\|_2 \neq 0)$.

$$\lambda \|v\|_2^2 = \langle \lambda v, v\rangle = \langle Av, v\rangle = \langle v, Av\rangle = \langle v, \lambda v\rangle = \lambda^* \|v\|_2^2$$

$\lambda = \lambda^* \Rightarrow \lambda \in \mathbb{R}$.

Now: show it's unitarily diagonalizable.

Strategy: induction

1.) Identify (λ, v) an eigenpair

2.) "Compress" A to an $(n-1)\times(n-1)$ matrix corresponds to the orthogonal complement of span$\{v\}$.

Details: $n=1$

$A = A^* \Rightarrow A \in \mathbb{R}$

$A = 1 \cdot A \cdot 1$ ✓

Inductive hypothesis: assume any Hermitian $B \in \mathbb{C}^{(n-1)\times(n-1)}$ is unitarily diagonalizable: $B = U \Lambda U^*$

$(U^* U = I, \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_{n-1}))$

Let $A = A^* \in \mathbb{C}^{n\times n}$.

Let (λ, v) be any eigenpair for A.

Without loss: take $\|v\|_2 = 1$.

Define $P := vv^*$ → this is the orthogonal projector onto span$\{v\}$.

$P^\perp := I - P$ → this is the orthogonal projector onto $W :=$ orthogonal complement of span$\{v\}$.

$$A = I \cdot A \cdot I = (P + P^\perp) A (P + P^\perp)$$

$$= \underset{①}{PAP} + \underset{②}{P^\perp A P^\perp} + \underset{③}{PAP^\perp} + \underset{④}{P^\perp AP}.$$

④: Let $x \in \mathbb{C}^n$

$$P^\perp APx = P^\perp A(cv) \quad (\text{for some } c \in \mathbb{C})$$

$$= cP^\perp \lambda v$$

$$= c\lambda P^\perp v = 0$$

$$\Rightarrow P^\perp AP = 0.$$

③ : same thing ($A=A^*$) implies $PAP^\perp = 0$.

①: $P = vv^*$

$$PAP = vv^*Avv^* = \lambda v \overbrace{v^*v}^{1} v^*$$

$$= \lambda vv^*$$

② $P^\perp A P^\perp$

$$P^\perp = QQ^*, \quad Q = \begin{pmatrix} | & & | \\ q_1 & \cdots & q_{n-1} \\ | & & | \end{pmatrix} \in \mathbb{C}^{n \times (n-1)}$$

where $\{q_j\}_{j=1}^{n-1}$ is an orthonormal basis for W.

$$P^\perp A P^\perp = Q \underbrace{Q^* A Q}_{\text{"}B\text{"} \in \mathbb{C}^{(n-1)\times(n-1)}} Q^* = QBQ^*$$

$B = B^*$ (since $A = A^*$)

inductive hypothesis: $B = U \Lambda U^*$, $U^*U = I$

$$P^\perp A P^\perp = Q B Q^* = \underbrace{Q U}_{\substack{S \in \mathbb{C}^{n\times(n-1)} \text{ has ON columns.} \\ (S^* S = I)}} \Lambda U^* Q^*$$

$$= S \Lambda S^*$$

Also: columns of S are orthogonal to v.
(range$(Q) = W$).

$$S = \begin{pmatrix} | & & | \\ s_1 & \cdots & s_{n-1} \\ | & & | \end{pmatrix} \longrightarrow S \Lambda S^* = \sum_{j=1}^{n-1} s_j s_j^* \lambda_j$$

$$\Lambda = \operatorname{diag}(\lambda_1 \ldots \lambda_{n-1}).$$

So: $P^\perp A P^\perp = \sum_{j=1}^{n-1} s_j s_j^* \lambda_j$

$$\Longrightarrow A = PAP + P^\perp A P^\perp = \lambda v v^* + \sum_{j=1}^{n-1} \lambda_j s_j s_j^*$$

$$= \tilde{U} \tilde{\Lambda} \tilde{U}^*$$

$$\tilde{U} = \begin{pmatrix} | & | & & | \\ v_1 & s_1 & \cdots & s_{n-1} \\ | & | & & | \end{pmatrix}, \quad \tilde{\Lambda} = \operatorname{diag}(\lambda, \lambda_1 \ldots \lambda_{n-1})$$

$$\tilde{U}^* \tilde{U} = I \qquad \square$$

A spectral theorem

Theorem
If $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ is Hermitian, then it is unitarily diagonalizable with real eigenvalues.

Hermitian matrices are also called *self-adjoint*. If $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ is unitarily diagonalizable, then it can be written as

matrix algebra

$$\boldsymbol{A} = \underline{\boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^*} = \sum_{j=1}^{n} \lambda_j \boldsymbol{u}_j \boldsymbol{u}_j^*,$$

where $\{\boldsymbol{u}_j\}_{j=1}^{n}$ are the columns of $\boldsymbol{U}$. (which are orthonormal).

$$Ax = \sum_{j=1}^{n} \lambda_j \underbrace{\langle x, u_j \rangle}_{\text{"amount" of } x \text{ points in } u_j} u_j.$$

"amount" of x points in u_j. direction

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where $\{\boldsymbol{u}_j\}_{j=1}^{n}$ are the columns of $\boldsymbol{U}$.

For example, the *spectral radius* of a matrix $\boldsymbol{A}$ is

$$\rho(\boldsymbol{A}) := \max_{j=1,\ldots,n} |\lambda_j(\boldsymbol{A})|$$

If $\boldsymbol{A}$ is Hermitian, then $\|\boldsymbol{A}\|_2 = \rho(\boldsymbol{A})$. (this is our first computable expression for $\|\cdot\|_2$).

(Hermitian) Positive-definite matrices

A matrix $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ is Hermitian positive definite (sometimes *symmetric* positive-definite or "spd") if it's Hermitian and its spectrum is strictly positive.

(Respectively, positive semi-definite if the spectrum is non-negative.)

A is spd $\Rightarrow \lambda(A) \in (0, \infty)$.

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Such matrices actually define a norm: $\|\boldsymbol{x}\|_{\boldsymbol{A}}^2 := \boldsymbol{x}^* \boldsymbol{A} \boldsymbol{x}$ is a norm.

$= \langle Ax, x \rangle = \langle x, Ax \rangle.$

(different from $\langle Ax, Ax \rangle$).

Matrix square roots

There is also a functional calculus on spd matrices.

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Example

If $\boldsymbol{A}$ is spd, compute a matrix square root of $\boldsymbol{A}$.

$$A = U \Lambda U^*, \quad \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \quad (A \in \mathbb{C}^{n \times n})$$

$$\lambda_j > 0 \quad \forall j.$$

$$\sqrt{\Lambda} := \mathrm{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}).$$

$$S = U \sqrt{\Lambda}\, U^* \Longrightarrow S \cdot S = U \Lambda U^* = A.$$

"$S = \sqrt{A}$"! (Note that $S = S^*$)

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Theorem

If $\boldsymbol{A}$ is spd, then there is a(n essentially) unique spd square root of $\boldsymbol{A}$.

(It's the one from the previous example.)

"essentially"? Up to unitary transformations of eigenspace E_λ with geometric multiplicity greater than 1.