Eigendecompositions of Hermitian matrices

MATH 6610 Lecture 03

September 4, 2020

Diagonalizability

Recall:

- All non-defective square matrices are diagonalizable (eigenvalue decomposition)
- All square matrices are bidiagonalizable (Jordan normal form)
- All square matrices are unitarily triangularizable (Schur decomposition)

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When are matrices unitarily diagonalizable?

A spectral theorem

Theorem

If $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ *is Hermitian, then it is unitarily diagonalizable with real eigenvalues.*

Hermitian matrices are also called *self-adjoint*.

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Hermitian matrices are also called *self-adjoint*. If $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ is unitarily diagonalizable, then it can be written as

$$\boldsymbol{A} = \boldsymbol{U} \boldsymbol{\Lambda} \boldsymbol{U}^* = \sum_{j=1}^{n} \lambda_j \boldsymbol{u}_j \boldsymbol{u}_j^*,$$

where $\{\boldsymbol{u}_j\}_{j=1}^n$ are the columns of $\boldsymbol{U}$.

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For example, the *spectral radius* of a matrix $\boldsymbol{A}$ is

$$\rho(\boldsymbol{A}) := \max_{j=1,\ldots,n} |\lambda_j(\boldsymbol{A})|$$

If $\boldsymbol{A}$ is Hermitian, then $\|\boldsymbol{A}\|_2 = \rho(\boldsymbol{A})$.

(Hermitian) Positive-definite matrices

A matrix $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ is Hermitian positive definite (sometimes *symmetric* positive-definite or "spd") if it's Hermitian and its spectrum is strictly positive.

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Such matrices actually define a norm: $\|\boldsymbol{x}\|_{\boldsymbol{A}}^2 := \boldsymbol{x}^* \boldsymbol{A} \boldsymbol{x}$ is a norm.

Matrix square roots

There is also a functional calculus on spd matrices.

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Theorem

If $\boldsymbol{A}$ is spd, then there is a(n essentially) unique spd square root of $\boldsymbol{A}$.