Eigenvalues and eigenvectors

MATH 6610 Lecture 02

September 2, 2020

Lecture 24 (Trefethen & Bau)

Eigenvalues and eigenvectors

Given $\boldsymbol{A} \in \mathbb{C}^{n \times n}$, $(\lambda, \boldsymbol{v}) \in \mathbb{C} \times (\mathbb{C}^n \backslash \{0\})$ is an eigenvalue-eigenvector pair if

$$\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}.$$

To find eigenvalues (on paper):
compute roots λ of $\det(A - \lambda I)$

To compute eigenvectors, solve $(A - \lambda I)v = 0$ for v.

Eigenpair properties

There are many properties of eigenvalues and eigenvectors of $\boldsymbol{A} \in \mathbb{C}^{n \times n}$. With $\lambda(\boldsymbol{A})$ the spectrum (collection of eigenvalues) of $\boldsymbol{A}$:

$\lambda(A) \subset \mathbb{C}$

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- All square matrices have exactly n eigenvalues, with some possibly repeated.

$\det(A - \lambda I)$ is a degree-n poly. in λ, so Fund. Theorem of Algebra, $\exists$ n complex-valued roots (possibly repeated).

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- All square matrices have at least 1 eigenvector.
- The collection of all eigenvectors associated to an eigenvalue λ is a subspace, and is frequently called an *eigenspace* E_λ.

(Pedantically: 0 is not an eigenvector, so set of all eigenvectors associated to λ cannot contain 0, so can't be a subspace.)

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- Eigenspaces are invariant subspaces of A.

$$A E_\lambda = E_\lambda$$

$$\shortparallel$$

$$\{ A x \mid x \in E_\lambda \}$$

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- The number of times an eigenvalue λ is repeated a_λ is its *algebraic multiplicity*
- The *geometric multiplicity* g_λ of an eigenvalue λ is $\dim E_\lambda$.

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad$ $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$a_1 = g_1 = 2$ $\qquad$ $a_1 = 2, \; g_1 = 1$

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- $1 \leqslant \sum_{\lambda \in \lambda(\boldsymbol{A})} g_\lambda \leqslant \sum_{\lambda \in \lambda(\boldsymbol{A})} a_\lambda = n$.

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- Eigenvalues λ with $g_\lambda < a_\lambda$ are *defective*

$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$\lambda = 1$ is defective.

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- $1 \leqslant \sum_{\lambda \in \lambda(\boldsymbol{A})} g_\lambda \leqslant \sum_{\lambda \in \lambda(\boldsymbol{A})} a_\lambda = n$.
- *Simple* eigenvalues λ have $g_\lambda = a_\lambda = 1$.
- Eigenvalues λ with $g_\lambda < a_\lambda$ are *defective*
- Any $\boldsymbol{A}$ such that $\sum_{\lambda \in \lambda(\boldsymbol{A})} g_\lambda < n$ is *defective*.

Similarity and Diagonalizability

Two square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are *similar* if $\exists$ an invertible $\boldsymbol{S}$ such that

$$\boldsymbol{B} = \boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S}.$$

(The map $\boldsymbol{A} \mapsto \boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S}$ is a similarity transform.)

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Definition

A square matrix $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ is <u>diagonalizable</u> if it is similar to a diagonal matrix.

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Theorem

A square matrix $\boldsymbol{A}$ is diagonalizable iff it is not defective.

Proof (approximately)

diagonal

if A is diagonalizable: $D = S^{-1} A S$

Ie. $D = S^{-1}AS$

$SD = AS$

$S = \begin{bmatrix} | & & | \\ s_1 & \cdots & s_n \\ | & & | \end{bmatrix}$

$\Rightarrow \begin{bmatrix} d_1 s_1 & \cdots & d_n s_n \end{bmatrix}$

$= \begin{bmatrix} A s_1 & \cdots & A s_n \end{bmatrix}$

this implies A not defective.

Other way: similar. ....

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Definition

A square matrix $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ is diagonalizable if it is similar to a diagonal matrix.

Theorem

A square matrix $\boldsymbol{A}$ is diagonalizable iff it is not defective.

When $\boldsymbol{A}$ is not defective, it is diagonalizable via a matrix whose columns are comprised of its linearly independent eigenvectors.

$AV = \Lambda V \quad , A = V^{-1}\Lambda V. \quad V = [v_1 \cdots v_n]$

$\Lambda = \mathrm{diag}(\lambda_1 \cdots \lambda_n).$

Similarity invariances

The set of eigenvalues is invariant under a similarity transform.

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This implies that if $\boldsymbol{A}$ is diagonalizable, then

$$\det \boldsymbol{A} = \prod_{j=1}^{n} \lambda_j, \qquad \mathrm{Tr}\boldsymbol{A} = \sum_{j=1}^{n} \lambda_j.$$

The above is actually true for any square matrix $\boldsymbol{A}$, defective or not.

Generalizations

While not all square matrices are diagonalizable...

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

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- ...all matrices are bidiagonalizable (Jordan normal form)

$A = V^{-1} J V$

J: bidiagonal (entries only on main and superdiagonal).

Generalizations

While not all square matrices are diagonalizable...

- ...all matrices are bidiagonalizable (Jordan normal form)
- ...all matrices are unitarily triangularizable (Schur decomposition)

similarity transform is unitary (U)

$A = U^* T U \qquad T = \begin{pmatrix} \diagdown & \\ 0 & \end{pmatrix}$

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When are matrices unitarily diagonalizable?