Eigenvalues and eigenvectors

MATH 6610 Lecture 02

September 2, 2020

Eigenvalues and eigenvectors

Given $\boldsymbol{A} \in \mathbb{C}^{n \times n}$, $(\lambda, \boldsymbol{v}) \in \mathbb{C} \times (\mathbb{C}^n \backslash \{0\})$ is an eigenvalue-eigenvector pair if

$$\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}.$$

Eigenpair properties

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- *Simple* eigenvalues λ have $g_\lambda = a_\lambda = 1$.
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- Any $\boldsymbol{A}$ such that $\sum_{\lambda \in \lambda(\boldsymbol{A})} g_\lambda < n$ is *defective*.

Similarity and Diagonalizability

Two square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are *similar* if $\exists$ an invertible $\boldsymbol{S}$ such that

$$\boldsymbol{B} = \boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S}.$$

(The map $\boldsymbol{A} \mapsto \boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S}$ is a similarity transform.)

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A square matrix $\boldsymbol{A}$ is diagonalizable iff it is not defective.

When $\boldsymbol{A}$ is not defective, it is diagonalizable via a matrix whose columns are comprised of its linearly independent eigenvectors.

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This implies that if $\boldsymbol{A}$ is diagonalizable, then

$$\det \boldsymbol{A} = \prod_{j=1}^{n} \lambda_j, \qquad \mathrm{Tr} \boldsymbol{A} = \sum_{j=1}^{n} \lambda_j.$$

The above is actually true for any square matrix $\boldsymbol{A}$, defective or not.

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When are matrices unitarily diagonalizable?