Projection and permutation matrices

MATH 6610 Lecture 01

August 31, 2020

"Fundamental Thm. of Linear Algebra"

$A \in \mathbb{C}^{m \times n}$

"4 fundamental subspaces (of A)"

1.) $\text{range}(A) \subset \mathbb{C}^m$

2.) $\ker(A) \subset \mathbb{C}^n$

3.) $\text{range}(A^*) \subset \mathbb{C}^n$

4.) $\ker(A^*) \subset \mathbb{C}^m$

Then: if $r = \text{rank}(A)$,

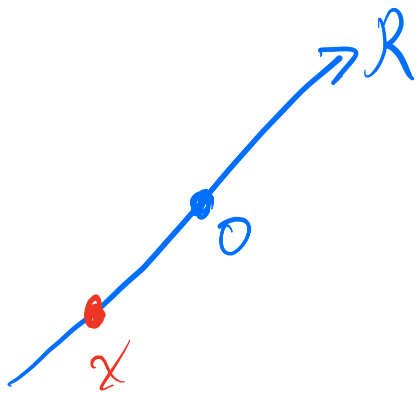
- $\dim \text{range}(A) = \dim \text{range}(A^*) = r$
- $\dim \ker(A) = n - r$
- $\dim \ker(A^*) = m - r$
- $\text{range}(A) \perp \ker(A^*)$
- $\text{range}(A^*) \perp \ker(A)$

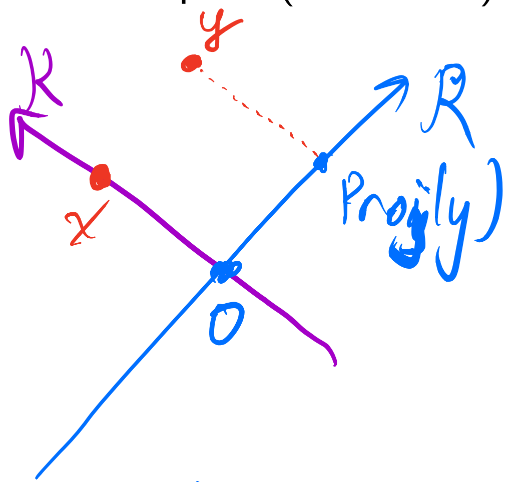
($S_1 \perp S_2$ if $\langle x, y \rangle = 0 \quad \forall\, x \in S_1,\ y \in S_2$)

Projections

With $\mathbb{C}^n$ the ambient space, we want to define "projections". Informally, we want projections to

- Act like the identity on some subspace (the range) (R)
- Annihilate components in another subspace (the kernel) (K)





$\mathrm{Proj}(x) = x$ $\qquad$ $\mathrm{Proj}(x) = 0$

Projections

With $\mathbb{C}^n$ the ambient space, we want to define "projections". Informally, we want projections to

- Act like the identity on some subspace (the range)
- Annihilate components in another subspace (the kernel)

In order to be well-defined, we need an additional condition.

Definition

A matrix $\boldsymbol{P} \in \mathbb{C}^{n \times n}$ is a projection matrix if

(!) ✓

1. $\boldsymbol{P}\boldsymbol{v} = \boldsymbol{v}$ for all $\boldsymbol{v} \in \mathrm{range}(\boldsymbol{P})$
2. $\boldsymbol{P}\boldsymbol{v} = \boldsymbol{0}$ for all $\boldsymbol{v} \in \ker(\boldsymbol{P})$
3. $\mathrm{range}(\boldsymbol{P}) \oplus \ker(\boldsymbol{P}) = \mathbb{C}^n$. $\Rightarrow \forall\, v \in \mathbb{C}^n, \exists\, r \in \mathrm{range}(P), k \in \ker(P)$ s.t. $v = r + k$.

Ex. $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \text{range}(P) = \mathbb{C}^2$

$\ker(P) = \{0\}$

$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad \text{range}(P) = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$

$\ker(P) = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$

$P = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

$P = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ is <u>not</u> a projection.

Projections exist and are unique

Does this definition make sense?

Theorem

If $\mathcal{R}$ and $\mathcal{K}$ are $\mathbb{C}^n$-subspaces such that $\mathcal{R} \cap \mathcal{K} = \overset{\{0\}}{\cancel{\varnothing}}$ and $\dim \mathcal{R} + \dim \mathcal{K} = n$, then $\exists\ !$ projection matrix $\boldsymbol{P} \in \mathbb{C}^{n \times n}$ such that $\mathrm{range}(\boldsymbol{P}) = \mathcal{R}$ and $\ker(\boldsymbol{P}) = \mathcal{K}$.

Proof: $\left.\begin{array}{l} \dim R + \dim K = n \\ R \cap K = \{0\} \end{array}\right\} \Longrightarrow R \oplus K = \mathbb{C}^n$

Existence: Let $r = \dim R$ $(0 < r < n)$

Let $\{a_j\}_{j=1}^{r}$ be a basis for R.

Let $\{b_j\}_{j=1}^{r}$ be a basis for $K^\perp$

$K^\perp = \{x \in \mathbb{C}^n \mid \langle x, y \rangle = 0 \ \forall y \in K\}$.

Construct $A = \begin{pmatrix} | & & | \\ a_1 & \cdots & a_r \\ | & & | \end{pmatrix}$, $B = \begin{pmatrix} | & & | \\ b_1 & \cdots & b_r \\ | & & | \end{pmatrix}$.

$A, B \in \mathbb{C}^{n \times r}$

Define $P = A\underbrace{(B^* A)}_{r \times r}{}^{-1} B^*$ ☆

if B^*A is invertible:

- if $x \in K$, then $B^* x = 0 \Rightarrow Px = 0$
- if $x \in R$, then $\exists y \in \mathbb{C}^r$ s.t. $x = Ay$

$$Px = A(B^*A)^{-1}B^*Ay$$
$$= Ay = x$$

is B^*A invertible?

- suppose $\exists x \in \mathbb{C}^r$ s.t. $B^*Ax = 0$.

$$\Rightarrow Ax \in K$$

But also: $Ax \in R$ ($\text{range}(A) = R$)

I.e. $Ax \in R \cap K$.

$\Rightarrow Ax = 0$.

But since columns of A are a basis for $R \Rightarrow Ax = 0 \Rightarrow x = 0$.

- the only solution to $B^*Ax = 0$ is $x = 0$.

$\Rightarrow B^*A$ invertible.

$\Rightarrow P$ is a projection.

uniqueness: Let P_1 and P_2 be projections with range R and kernel K.

$P_1 = P_2 \Leftrightarrow (P_1 - P_2)v = 0 \quad \forall v \in \mathbb{C}^n$.

Let $v = v_R + v_K$, $v_R \in R$, $v_K \in K$.

$$(P_1 - P_2)v = (P_1 - P_2)(v_R + v_K) = (P_1 - P_2)v_R$$
$$= v_R - v_R = 0.$$

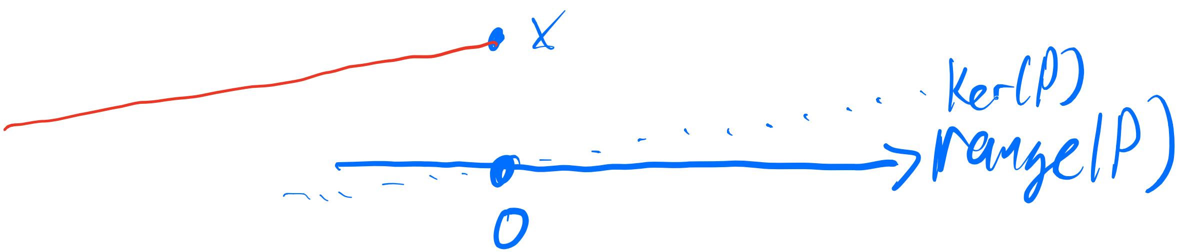
$\Rightarrow$ uniqueness

Orthogonal projections

Projection matrices can in general inflate the size (norm) of non-trivial vectors by an arbitrary amount.

If P is a projection

$$\Longrightarrow \|P\|_2 = \sup_{x \in \mathbb{C}^n \setminus \{0\}} \frac{\|Px\|_2}{\|x\|_2} \geq \sup_{x \in \mathrm{range}(P) \setminus \{0\}} \frac{\|Px\|_2}{\|x\|_2} = 1$$



Orthogonal projections

Projection matrices can in general inflate the size (norm) of non-trivial vectors by an arbitrary amount.

However, projecting along $(\mathrm{range} \boldsymbol{P})^\perp$ is a norm non-expansive operation, i.e.,

$$\ker(\boldsymbol{P}) = \mathrm{range}(\boldsymbol{P})^\perp \quad \Longrightarrow \quad \|\boldsymbol{P}\boldsymbol{v}\|_2 \leqslant \|\boldsymbol{v}\|_2.$$

$\Downarrow$

$\|P\|_2 = 1$

Pythagorean thm.

Orthogonal projections

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Projection matrices $\boldsymbol{P}$ satisfying $\ker(\boldsymbol{P}) = (\mathrm{range}\boldsymbol{P})^\perp$ are *orthogonal* projections.

Projection matrices

There is a more algebraically convenient characterization of projection matrices.

A square matrix $\boldsymbol{A}$ is *idempotent* if $\boldsymbol{A} = \boldsymbol{A}^2$.

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Theorem

$\boldsymbol{P} \in \mathbb{C}^{n \times n}$ *is a projection matrix iff it is idempotent.* $(P = P^2)$

Proof: Suppose P is a projection.

Let $v \in \mathbb{C}^n$. Then $v = v_R + v_K$

range(P) ker(P)

$$(P-P^2)v = (P-P^2)(v_R + v_K)$$

$$= (P-P^2)v_R = v_R - Pv_R = v_R - v_R = 0.$$

$$\Rightarrow P = P^2$$

Now suppose that $P = P^2$.

Let $v \in \mathbb{C}^n$.

$$v = \underbrace{Pv}_{\text{range}(P)} + \overbrace{(I-P)v}^{\ker(P)}$$

$$\big(P(I-P)v = (P-P^2)v = 0\big)$$

already know: $\dim \text{range}(P) + \dim \ker(P) = n$.

$\Rightarrow \text{range}(P) \oplus \ker(P) = \mathbb{C}^n$.

exercise: show $Pv = v \ \forall \ v \in \text{range}(P)$. $\square$

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Theorem

$\boldsymbol{P} \in \mathbb{C}^{n \times n}$ *is a projection matrix iff it is idempotent.*

This motivates the more common definition of a projection matrix:

Definition

$\boldsymbol{P} \in \mathbb{C}^{n \times n}$ is a projection matrix if $\boldsymbol{P} = \boldsymbol{P}^2$.

Orthogonal projectors

There is also an algebraically convenient characterization of orthogonal projection matrices.

Theorem

Let $\boldsymbol{P} \in \mathbb{C}^{n\times n}$ be a projection matrix. Then it is an orthogonal projector iff it is Hermitian.

$P = P^*$

projector

P is orthogonal if range$(P) \perp$ ker(P).

Proof: Assume $P = P^*$.

Two ways (i) range$(P) \perp$ ker(P^*) (FT of LA)

$\Longrightarrow$ range$(P) \perp$ ker(P).

(ii) Let $x \in \text{range}(P)$

$y \in \ker(P)$

$$\langle x, y\rangle = \langle Px, y\rangle = \langle Px, (I-P)y\rangle$$
$$= \langle x, P^*(I-P)y\rangle$$
$$= \langle x, \underbrace{(P^* - P^*P)}_{P-P^2=0.}y\rangle$$

$$= 0.$$

Now assume $\text{range}(P) \perp \ker(P)$

Show $\forall\, x, y \in \mathbb{C}^n : \langle Px, y\rangle = \langle P^* x, y\rangle$

$$\left.\begin{array}{l} x = x_R + x_K \\ y = y_R + y_K \end{array}\right\} \begin{array}{l} x_R, y_R \in \text{range}(P) \\ x_K, y_K \in \ker(P) \end{array}$$

$$\langle Px, y\rangle = \langle x_R, y_R + y_K\rangle = \langle x_R, y_R\rangle$$
$$\langle P^* x, y\rangle = \langle x, Py\rangle = \langle x_R, y_R\rangle$$
$$\Rightarrow P = P^* \quad \square.$$

Orthogonal projectors

There is also an algebraically convenient characterization of orthogonal projection matrices.

Theorem

Let $\boldsymbol{P} \in \mathbb{C}^{n \times n}$ be a projection matrix. Then it is an orthogonal projector iff it is Hermitian.

This motivates the more common definition of an orthogonal projection matrix:

Definition

$\boldsymbol{P} \in \mathbb{C}^{n \times n}$ is an orthogonal projection matrix if $\boldsymbol{P} = \boldsymbol{P}^2$ and $\boldsymbol{P} = \boldsymbol{P}^*$.

Note: if P is an orthogonal projector, then
$\ker(P)^{\perp} = \operatorname{range}(P)$.
If $q_1 \sim q_r$ is an orthonormal basis for $\operatorname{range}(P)$, ...

then $P = A(B^*A)^{-1}B^*$, $\quad$ $\text{range}(A) = \text{range}(P)$

$\text{range}(B) = \ker(P)^\perp$.

choose $A = B = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_r \\ | & & | \end{bmatrix} = Q$.

$B^*A = I_{r\times r}$.

$\Rightarrow \underline{P = QQ^*}$

$Px = Q\underbrace{Q^*x}_{\text{scalar projections of } x \text{ in directions } q_1 \cdots q_r.}$

Permutation

A second class of matrices we'll consider are permutation matrices.

Definition

For a fixed $n \in \mathbb{N}$, $\sigma : [n] \to [n]$ is a <u>permutation</u> if it is a bijection.

$[n] = \{1, 2, \ldots n\}$

Permutation

A second class of matrices we'll consider are permutation matrices.

Definition

For a fixed $n \in \mathbb{N}$, $\sigma : [n] \to [n]$ is a permutation if it is a bijection.

Definition

$\boldsymbol{P} \in \mathbb{C}^{n \times n}$ is a permutation matrix if there is a permutation map σ on $[n]$ such that $\boldsymbol{P}\boldsymbol{e}_j = \boldsymbol{e}_{\sigma(j)}$ for all $j \in [n]$.

$e_j = (0, 0 \cdots 0, 1, 0, \cdots 0, 0)^*$

jth element.