Projection and permutation matrices

MATH 6610 Lecture 01

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Projections

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- Annihilate components in another subspace (the kernel)

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In order to be well-defined, we need an additional condition.

Definition

A matrix $\boldsymbol{P} \in \mathbb{C}^{n \times n}$ is a projection matrix if

1. $\boldsymbol{P}\boldsymbol{v} = \boldsymbol{v}$ for all $\boldsymbol{v} \in \mathrm{range}(\boldsymbol{P})$
2. $\boldsymbol{P}\boldsymbol{v} = \boldsymbol{0}$ for all $\boldsymbol{v} \in \ker(\boldsymbol{P})$
3. $\mathrm{range}(\boldsymbol{P}) \oplus \ker(\boldsymbol{P}) = \mathbb{C}^n$.

Projections exist and are unique

Does this definition make sense?

Theorem

If $\mathcal{R}$ and $\mathcal{K}$ are $\mathbb{C}^n$-subspaces such that $\mathcal{R} \cap \mathcal{K} = \varnothing$ and $\dim \mathcal{R} + \dim \mathcal{K} = n$, then $\exists\ !$ projection matrix $\boldsymbol{P} \in \mathbb{C}^{n \times n}$ such that $\mathrm{range}(\boldsymbol{P}) = \mathcal{R}$ and $\ker(\boldsymbol{P}) = \mathcal{K}$.

Orthogonal projections

Projection matrices can in general inflate the size (norm) of non-trivial vectors by an arbitrary amount.

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Projection matrices $\boldsymbol{P}$ satisfying $\ker(\boldsymbol{P}) = (\mathrm{range}\boldsymbol{P})^\perp$ are *orthogonal* projections.

Projection matrices

There is a more algebraically convenient characterization of projection matrices.

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This motivates the more common definition of a projection matrix:

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Orthogonal projectors

There is also an algebraically convenient characterization of orthogonal projection matrices.

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Let $\boldsymbol{P} \in \mathbb{C}^{n \times n}$ *be a projection matrix. Then it is an orthogonal projector iff it is Hermitian.*

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Let $\boldsymbol{P} \in \mathbb{C}^{n \times n}$ be a projection matrix. Then it is an orthogonal projector iff it is Hermitian.

This motivates the more common definition of an orthogonal projection matrix:

Definition

$\boldsymbol{P} \in \mathbb{C}^{n \times n}$ is an orthogonal projection matrix if $\boldsymbol{P} = \boldsymbol{P}^2$ and $\boldsymbol{P} = \boldsymbol{P}^*$.

Permutation

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Definition

$\boldsymbol{P} \in \mathbb{C}^{n \times n}$ is a permutation matrix if there is a permutation map σ on $[n]$ such that $\boldsymbol{P}\boldsymbol{e}_j = \boldsymbol{e}_{\sigma(j)}$ for all $j \in [n]$.