Linear algebra preliminaries

MATH 6610 Lecture 00

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Trefethen & Bau Lectures 1-3.

Notation

We'll use some standard math notation

- $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$ (integers)
- $\in$, $\forall$, $\exists$, !
- $\{x \in \mathbb{C} \mid \mathrm{Im}\{x\} \in \mathbb{N}\}$
- $z = x + iy$ for $x, y \in \mathbb{R} \Longrightarrow \bar{z} = z^* := x - iy$

Vectors, matrices, etc:

- $\boldsymbol{u} \in \mathbb{C}^n$
- $\boldsymbol{A} \in \mathbb{C}^{m \times n}$

$(\underline{u}, \underline{\underline{A}})$

- linear independence
- rank
- transpose
- determinant
- matrix inverse

$\mathrm{rank}(A) = \mathrm{colrank}(A) = \#$ of LI columns

$= \mathrm{rowrank}(A) = \#$ of LI rows

$A^* \in \mathbb{C}^{n \times m}$

$(A^*)_{i,j} = (A_{j,i})^*$

Hilbertian structure

$\mathbb{C}^n$ endowed with the standard inner product is a Hilbert space. If $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{C}^n$,

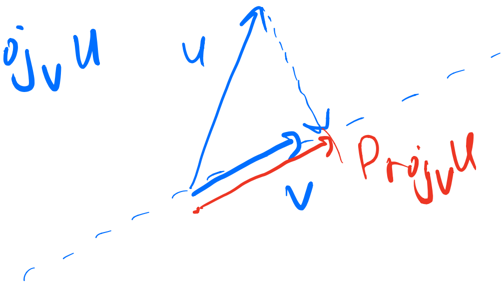
- $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$, $\|\boldsymbol{u}\|$
- $\angle(\boldsymbol{u}, \boldsymbol{v})$
- $\boldsymbol{u} \perp \boldsymbol{v}$
- $\mathrm{Proj}_{\boldsymbol{v}} \boldsymbol{u}$
- orthogonal and orthonormal sets

$\langle u, v \rangle = \sum_{j=1}^{n} u_j v_j^*$, $\|u\|^2 := \langle u, u \rangle$

$\angle(u,v) = \arccos\left(\frac{\langle u, v\rangle}{\|u\| \cdot \|v\|}\right)$, $\arccos : [-1,1] \to [0, \pi]$

$u \perp v \iff \langle u, v \rangle = 0$ $(u, v \neq 0)$

$\text{Proj}_v u$



$$\text{Proj}_v u = \underbrace{\frac{\langle u, v \rangle}{\|v\|}}_{\text{"amount" of } u \text{ pointing in } v \text{ direction}} \frac{v}{\|v\|} \quad (v \neq 0)$$

$\{u_j\}_{j=1}^{m} \subset \mathbb{C}^n$ are orthogonal if $\langle u_j, u_k \rangle = 0$ if $j \neq k$

" orthonormal if $\langle u_j, u_k \rangle = \delta_{j,k} = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$

Unitary and orthogonal matrices

We'll frequently let $\boldsymbol{U} \in \mathbb{C}^{n \times n}$ denote a unitary/orthogonal matrix.

Def: $U \in \mathbb{C}^{n \times n}$ is unitary if its columns are orthonormal.

($U \in \mathbb{R}^{n \times n}$ is unitary, it's sometimes called orthogonal)

$U^* U = I_{n \times n} \Longrightarrow U^{-1} = U^* \Longrightarrow U U^* = I_{n \times n}$

$\Longrightarrow$ rows of U are orthonormal.

Ex. $\|Ux\| = \|x\|$ (HW)

Vector norms

A map $\|\cdot\| : \mathbb{C}^n \to \mathbb{R}$ is a *norm* if it satisfies all the following properties:

- $\|\boldsymbol{x}\| \geqslant 0 \ \ \forall \boldsymbol{x} \in \mathbb{C}^n$
- $\|\boldsymbol{x}\| = 0$ iff $\boldsymbol{x} = \mathbf{0}$
- $\|\boldsymbol{x} + \boldsymbol{y}\| \leqslant \|\boldsymbol{x}\| + \|\boldsymbol{y}\| \ \ \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^n$ ("triangle inequality")
- $\|c\boldsymbol{x}\| = |c|\|\boldsymbol{x}\| \ \ \forall \boldsymbol{x} \in \mathbb{C}^n, \ c \in \mathbb{C}$.

For example there are vector p-norms.

(Above $\|\cdot\|$ does not necessarily mean Euclidean norm)

Ex (p-norms)

$x \in \mathbb{C}^n$, $\quad \|x\|_p := \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p}, \quad 1 \le p < \infty$

$$\|x\|_\infty := \max_{1 \le j \le n} |x_j|$$

Euclidean norm: $p = 2$

There are many properties of (p-) norms:

Ex. (Cauchy-Schwarz)

$x, y \in \mathbb{C}^n \quad |\langle x, y\rangle| \leq \|x\|_2 \|y\|_2$

Ex. (Hölder inequalities)

Let $p, q \in (1, \infty)$ be "conjugate exponents", i.e.

$\frac{1}{p} + \frac{1}{q} = 1$.

Then: $|\langle x, y\rangle| \leq \|x\|_p \|y\|_q$

($p = q = 2 \Longrightarrow$ Cauchy-Schwarz)

How "different" are these norms?

Theorem ("All finite-dimensional norms are equivalent")

Given $\|\cdot\|_a, \|\cdot\|_b : \mathbb{C}^n \to [0, \infty)$ that are norms, then

$\exists\ c, C \in (0, \infty)$ such that

$$c\|x\|_a \leq \|x\|_b \leq C\|x\|_a \quad \forall x \in \mathbb{C}^n$$

(I.e. c, C depend n, $\|\cdot\|_a$, $\|\cdot\|_b$, but are independent of x).

Matrix norms

Norms on matrices can be *induced* by vector norms, but there are also non-induced norms.

Def: The induced p norm on matrices $A \in \mathbb{C}^{n \times n}$ is

$$\|A\|_p := \sup_{\substack{\cancel{x \in \mathbb{C}^n} \\ x \in \mathbb{C}^n \setminus \{0\}}} \frac{\|Ax\|_p}{\|x\|_p} = \sup_{\substack{x \in \mathbb{C}^n \\ \|x\|_p = 1}} \|Ax\|_p \qquad (1 \le p \le \infty)$$

scale invariance of norms

Note: supremum is achievable in $\mathbb{C}^n$

Def: Given $A \in \mathbb{C}^{m \times n}$, and given $p, q \in [1, \infty]$, then

$$\|A\|_{p,q} := \sup_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \frac{\|Ax\|_p}{\|x\|_q} \quad (\text{"induced } (p,q) \text{ norm"}).$$

Ex. $A \in \mathbb{C}^{m \times n}$, $U \in \mathbb{C}^{m \times m}$ is unitary.

Then $\|UA\|_2 = \|A\|_2$

Induced norms are "sub-multiplicative", i.e, if $A, B \in \mathbb{C}^{n \times n}$, then $\|AB\| \leq \|A\| \cdot \|B\|$ if $\|\cdot\|$ is an induced norm.

There are non-induced norms. The most useful example is the Frobenius norm: $A \in \mathbb{C}^{m \times n}$

$$\|A\|_F^2 := \sum_{i=1}^{m} \sum_{j=1}^{n} |A_{ij}|^2$$

Ie, this norm is "entrywise".

Q: $\|A\|_F \stackrel{?}{=} \sqrt{\mathrm{Tr}(A)}$ X

$$\|A\|_F^2 = \mathrm{Tr}(A^* A)$$

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norm(A) (induced 2-norm)

$\longrightarrow$ answer