Department of Mathematics, University of Utah

Analysis of Numerical Methods I

MATH 6610 – Section 001 – Fall 2020

Homework 2

The SVD and QR factorizations

Due Monday, October 5, 2020 by 11:59pm MT

Submission instructions:
Create a private repository on `github.com` named `math6610-homework-2`. Add your LaTeX source files and your Matlab/Python code and push to Github. To submit: grant me (username `akilnarayan`) write access to your repository.
You may grant me write access before you complete the assignment. I will not look at your submission until the due date+time specified above. If you choose this route, I will only grade the assignment associated with the last commit before the due date.
All commits timestamped after the due date+time will be ignored
All work in commits before the final valid timestamped commit will be ignored.
Problem assignment:
In all the following exercises, if a program in Matlab is requested, replace those instructions with a request to write a program in the language of your choice.

Trefethen & Bau III, Lecture 4: # 4.1 (c,d), 4.4
Trefethen & Bau III, Lecture 5: # 5.2, 5.4
Trefethen & Bau III, Lecture 6: # 6.1
Trefethen & Bau III, Lecture 7: # 7.3, 7.4, 7.5
Trefethen & Bau III, Lecture 10: # 10.1
Trefethen & Bau III, Lecture 12: # 12.3

Additional problems:

P1. Let $A \in \mathbb{C}^{n\times n}$. Show that if A is Hermitian positive semi-definite, then the eigenvalues of A and the singular values of A are the same.

P2. For a square matrix $A \in \mathbb{C}^{n\times n}$, recall that the spectral radius $\rho(A)$ is the maximum modulus of all eigenvalues of A.

(a) Prove that $\rho(A) \leq \sigma_1(A)$.

(b) Prove that if A is normal, then $\rho(A) = \sigma_1(A)$.

P3. (Pseudoinverse) Let $A \in \mathbb{C}^{m\times n}$ have rank r and *reduced* SVD

$$A = \widetilde{U}\widetilde{\Sigma}\widetilde{V}^*$$

The *Moore-Penrose pseudoinverse* of A is defined as

$$A^+ = \widetilde{V}\widetilde{\Sigma}^{-1}\widetilde{U}^*$$

Prove the following:

(a) If A is invertible (hence also square), then $A^{-1} = A^+$.

(b) Prove that the matrices AA^+, A^+A, $I - AA^+$, and $I - A^+A$ are all projection matrices. Are they orthogonal? How would you characterize their ranges and kernels in terms of the fundamental subspaces defined by the matrix A?

(c) Give and prove conditions on A so that $AA^+A = A$.

(d) Show that $\|A^+\|_2 = 1/\sigma_r(A)$

(e) In the induced matrix ℓ^2 norm $\|\cdot\|$, is the operation $A \mapsto A^+$ well-conditioned? That is, given an arbitrary but fixed A and a perturbation matrix B, is $||(A+B)^+ - A^+||/||A^+||$ controllable by $||B||/||A||$? Prove it, or give a counterexample.

P4. Let $A \in \mathbb{C}^{m\times n}$ and $b \in \mathbb{C}^n$ be given. Show that $x = A^+b$ solves

$$\min_{z\in\mathbb{C}^n} ||z||_2 \quad \text{subject to} \quad \|Az - b\|_2 \text{ is minimized,}$$

What is the difference between this x and the linear least-squares solution to $Az = b$?

P5. (ϵ-pseudospectrum) Let $A \in \mathbb{C}^{n\times n}$. We let $\Lambda(A)$ denote the spectrum (set of eigenvalues) of A. For some $\epsilon > 0$, the ϵ-pseudospectrum of A is the set of complex numbers $\Lambda_\epsilon(A)$ given by

$$\Lambda_\epsilon(A) \coloneqq \left\{ z \in \mathbb{C} \;\middle|\; z \in \Lambda(A + \Delta A) \text{ for some } \Delta A \in \mathbb{C}^{n\times n} \text{ satisyfing } \|\Delta A\|_2 \le \epsilon \right\}.$$

Another concept we'll need is the *resolvent*. The resolvent of A at $z \in \mathbb{C}$ is defined as the matrix $\mathrm{Res}_A(z) \coloneqq (z\,I - A)^{-1}$.

Given $\epsilon > 0$ and $A \in \mathbb{C}^{n\times n}$, prove that the following three sets of numbers in the complex plane are identical:

(a) $\Lambda_\epsilon(A)$

(b) The set of $z \in \mathbb{C}$ satisfying $\|(A - zI)v\|_2 \le \epsilon$ for some unit vector v.

(c) $\left\{ z \in \mathbb{C} \;\middle|\; \|\mathrm{Res}_A(z)\|_2 \ge 1/\epsilon \right\}$, with the convention that $\|\mathrm{Res}_A(z)\|_2 = \infty$ if $z \in \Lambda(A)$.

P6. (Bauer-Fike eigenvalue perturbations) Suppose $A \in \mathbb{C}^{n\times n}$ is diagonalizable with eigenvalue decomposition $A = V\Lambda V^{-1}$. Let ΔA be any matrix satisfying $\|\Delta A\|_2 \le \epsilon$. Suppose $\tilde{\lambda}$ is an(y) eigenvalue of $A + \Delta A$. Show that there is some eigenvalue $\lambda \in \Lambda(A)$ such that

$$\left|\lambda - \tilde{\lambda}\right| \le \epsilon\, \kappa(V),$$

with $\kappa(V)$ the (2-norm) condition number of V. The typical way to prove this is to exploit the equivalence between the sets of complex numbers in P5. (E.g., rewrite the resolvent with the knowledge that A is diagonalizable.). If A is unitarily diagonalizable, what does the bound above simplify to?

P7. (Schur Decomposition) Prove the Schur decomposition: every $A \in \mathbb{C}^{n\times n}$ is unitarily triangularizable. I.e., there exists a square unitary matrix Q and an upper triangular matrix U such that $A = QUQ^*$. (The main idea is induction on n. In the inductive step, apply to A a similarity transform using a unitary matrix $\widetilde{Q}$ whose first columns are an orthonormal basis for any eigenspace, and the rest of the columns span the orthogonal complement.)

Computing assignment:

C1. (Linear dimensional reduction) In a programming language of your choice, plot SVD-compressed representations of the Yale Face Database:
`http://vision.ucsd.edu/content/yale-face-database`
For this problem, you will consider two experiments:

Experiment A. For this experiment, grayscale images can be viewed as a matrix A of numbers, where the dimensions of the matrix correspond to the pixel dimensions of the image. Approximate an image A by a rank-r SVD approximation. (Pick a couple of your favorites.) When plotting the original images versus the approximations, how does the accuracy depend on r? Does the accuracy qualitatively depend on what kind of image (face) you use?

Experiment B. In order to treat an image as a single data point, vectorize its matrix representation (i.e., unwind the matrix into a vector of length m). For an experiment with n data points (images), let $A \in \mathbb{R}^{m \times n}$ be the resulting matrix. Use the rank-r SVD compression of A, each column of this rank-r matrix (appropriately unwound) is an approximation of one of the images. Explore the accuracy of rank-r compression for some values of r (experiment!). How does the qualitative accuracy of this experiment compare to the previous one?

In the above, design your own experiments: from all the images in a Yale database, select a couple of sets of them (maybe all of them, maybe only the ones with glasses, etc), and investigate the accuracy of rank-r approximation.

Such an SVD-based approach is called linear dimension reduction, since it's the (2-norm) optimal procedure for approximating data by *linear* projections. It's a reduction scheme since a rank-r matrix requires only $\mathcal{O}(r \max(m, n))$ storage instead of, e.g., $\mathcal{O}(m^2)$ storage, which is a substantial difference if $r \ll n, m$.

Submit your code, but <u>not</u> the Yale Face Database data files. In your code provide an easy way for me to point to a particular folder on my local hard drive that contains the data.