Department of Mathematics, University of Utah

Anaylsis of Numerical Methods I

MATH 6610 – Section 001 – Fall 2020

Homework 0

Basics of submission

Due Wednesday, September 2, 2020 by 11:59pm MT

Submission instructions:
Create a private repository on `github.com` named `math6610-homework-0`. Add your LaTeX source files and your Matlab/Python code and push to Github. To submit: grant me (username `akilnarayan`) write access to your repository.
You may grant me write access before you complete the assignment. I will not look at your submission until the due date+time specified above. If you choose this route, I will only grade the assignment associated with the last commit before the due date.
<u>All commits timestamped after the due date+time will be ignored</u>
<u>All work in commits before the final valid timestamped commit will be ignored.</u>

<u>Problem 1.</u>
Write a program that uses a Monte Carlo technique to estimate π.

Let $\rho : \mathbb{R}^d \to \mathbb{R}$ be a given probability density and let $f : \mathbb{R}^d \to \mathbb{R}$ be a given function. If X is a random variable distributed according to ρ, then a Monte Carlo estimate of $\mathbb{E}[f(X)]$ using a size-M ensemble is the approximation,

$$\mathbb{E}[f(X)] \coloneqq \int_{\mathbb{R}^d} f(x)\rho(x)\,\mathrm{d}x \approx F_M \eqqcolon \frac{1}{M}\sum_{m=1}^{M} f(X_m), \tag{1}$$

where $(X_m)_{m=1}^M$ are independent random variables distributed according to the density ρ. Note that F_M itself is a random variable.

Let $R = \left\{(x_1, x_2) \in \mathbb{R}^2 \,\middle|\, x_1^2 + x_2^2 \leq 1\right\}$ be the unit ball in $\mathbb{R}^2$. Define ρ as the uniform (probability) density on the square $[-1, 1]^2 \subset \mathbb{R}^2$, i.e., $\rho(x_1, x_2) = 1/4$ if both $|x_1|$ and $|x_2|$ are at most 1, and $\rho(x_1, x_2) = 0$ otherwise. Define $f(x)$ as the characteristic function of R, i.e.,

$$f(x) = \mathbb{1}_R(x) = \begin{cases} 1, & x \in R, \\ 0, & x \notin R \end{cases}$$

Since

$$\mathbb{E}[f(X)] = \int_{[-1,1]^2} f(x)\frac{1}{4}\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{4}\int_R \mathrm{d}x\,\mathrm{d}y = \frac{\pi}{4},$$

then F_M defined in (1) is an estimator for $\pi/4$.

Using your programming language of choice, write a program that computes $4F_M$ and hence estimates. (Note that from (1), to generate one instance of F_M you need only the ability to generate instances of the uniform random variable X, and to implement the function f.) Report the convergence of $4F_M$ to π as a function of M. Since F_M is random, you should report ensemble statistics of F_M (e.g., the deviation of a computed mean from π, an ensemble-computed standard deviation, etc.) Use your code to verify the central limit theorem for a sufficiently large M.

Note: This is a special assignment. The solution is already available to you: `https://github.com/akilnarayan/math6610-homework-0/`. Your task in this assignment is simply to practice submission. Clone the above repository, change the name on the document from mine to yours, and submit through Github. No actual mathematics or coding is required.