

Section 10.3 # 1, 2, 3, 5, 6, 11, 7

10.3.1

$$\mathcal{F}[f(x)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx$$

$$F(\omega) = \mathcal{F}[f(x)], \quad G(\omega) = \mathcal{F}[g(x)]$$

$$\begin{aligned} (a) \quad \mathcal{F}[c_1 f(x) + c_2 g(x)] &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (c_1 f(x) + c_2 g(x)) e^{i\omega x} dx \\ &= c_1 \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx + c_2 \frac{1}{2\pi} \int_{-\infty}^{\infty} g(x) e^{i\omega x} dx \\ &= c_1 \mathcal{F}[f(x)] + c_2 \mathcal{F}[g(x)] \end{aligned}$$

$$(b) \quad \mathcal{F}[f(x)g(x)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)g(x) e^{i\omega x} dx$$

$$F(\omega) \cdot G(\omega) = \frac{1}{4\pi^2} \left[\int_{-\infty}^{\infty} f(x) e^{i\omega x} dx \right] \cdot \left[\int_{-\infty}^{\infty} g(x) e^{i\omega x} dx \right]$$

These are not the same expression: $\mathcal{F}[f(x)g(x)] \neq F(\omega)G(\omega)$

10.3.2

$$\mathcal{F}^{-1}[F(\omega)] = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} d\omega$$

$$\begin{aligned} (a) \quad \mathcal{F}^{-1}[c_1 F(\omega) + c_2 G(\omega)] &= \int_{-\infty}^{\infty} (c_1 F(\omega) + c_2 G(\omega)) e^{-i\omega x} d\omega \\ &= c_1 \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} d\omega + c_2 \int_{-\infty}^{\infty} G(\omega) e^{-i\omega x} d\omega \\ &= c_1 \mathcal{F}^{-1}[F(\omega)] + c_2 \mathcal{F}^{-1}[G(\omega)] \end{aligned}$$

$$(b) \quad \mathcal{F}^{-1}[F(\omega)G(\omega)] = \int_{-\infty}^{\infty} F(\omega)G(\omega) e^{-i\omega x} d\omega$$

$$f(x)g(x) = \left[\int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} d\omega \right] \cdot \left[\int_{-\infty}^{\infty} G(\omega) e^{-i\omega x} d\omega \right]$$

These are not the same expression: $\mathcal{F}^{-1}[F(\omega)G(\omega)] \neq f(x)g(x)$

10.3.3 $F(\omega) = \mathcal{F}[f(x)]$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx \quad (e^{ix} = \cos x + i \sin x)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \cos x dx + \frac{i}{2\pi} \int_{-\infty}^{\infty} f(x) \sin x dx$$

so $F^*(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \cos x dx - \frac{i}{2\pi} \int_{-\infty}^{\infty} f(x) \sin x dx$ (since $f(x)$ is real,
 $f^*(x) = f(x)$)

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i(-\omega)x} dx = F(-\omega)$$

10.3.5 $F(\omega) = \mathcal{F}[f(x)]$

$$f(x) = \mathcal{F}^{-1}[F(\omega)] = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} d\omega$$

$$\mathcal{F}^{-1}[e^{i\omega\beta} F(\omega)] = \int_{-\infty}^{\infty} F(\omega) e^{i\omega\beta} e^{-i\omega x} d\omega$$
$$= \int_{-\infty}^{\infty} F(\omega) e^{-i\omega(x-\beta)} d\omega$$
$$= f(x-\beta)$$

10.3.6 $f(x) = \begin{cases} 0, & |x| > a \\ 1, & |x| < a \end{cases}$

Note: $e^{ix} = \cos x + i \sin x$
 $\Rightarrow \sin x = \frac{1}{2i} [e^{ix} - e^{-ix}]$

$$\mathcal{F}[f(x)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx = \frac{1}{2\pi} \int_{-a}^a 1 \cdot e^{i\omega x} dx$$

$$= \frac{1}{2\pi i \omega} e^{i\omega x} \Big|_{x=-a}^a$$

$$= \frac{1}{2\pi i \omega} [e^{i\omega a} - e^{-i\omega a}] = \frac{\sin(\omega a)}{\pi \omega}$$

10.3.11 (a) Assume $f(x)$ is a function with unit area, i.e.

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

Then consider, for $\alpha > 0$,

$$\int_{-\infty}^{\infty} \frac{1}{\alpha} f(x/\alpha) dx = \int_{-\infty}^{\infty} f(u) du = 1$$

$$u = x/\alpha$$

$$du = \frac{1}{\alpha} dx$$

So $\frac{1}{\alpha} f(x/\alpha)$ also has unit area.

(b) $F(\omega) = \mathcal{F}[f(x)]$

$$F(\alpha\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\alpha\omega x} dx = \frac{1}{2\pi} \cdot \frac{1}{\alpha} \int_{-\infty}^{\infty} f(u/\alpha) e^{i\omega u} du$$

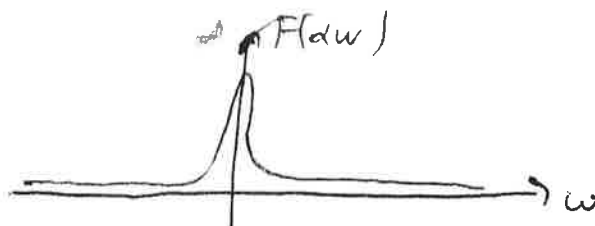
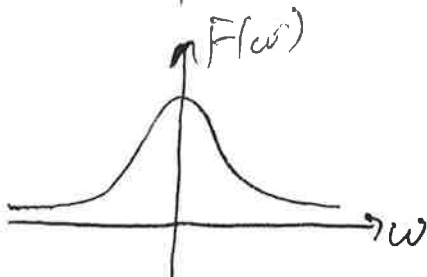
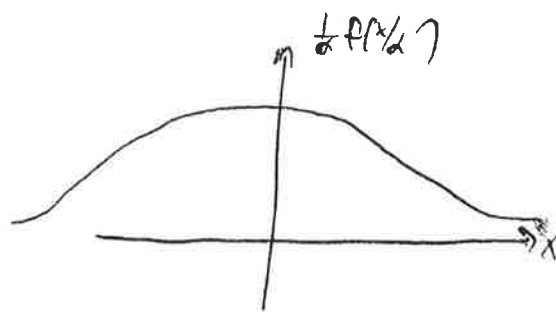
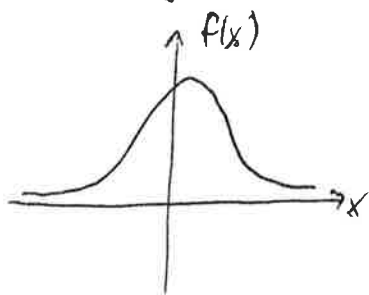
$$u = \alpha x$$

$$du = \alpha dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{1}{\alpha} f(u/\alpha) \right] e^{i\omega u} du$$

$$= \mathcal{F}\left[\frac{1}{\alpha} f(x/\alpha) \right].$$

(c) Consider large α :



Thus spread-out functions $(\frac{1}{\alpha} f(x/\alpha))$ have sharply-peaked Fourier transforms $(F(\alpha\omega))$. Taking small values of α shows the reverse relation: sharply-peaked functions have spread-out Fourier transforms.

10.3.7 $F(w) = e^{-|w|a}$, $a > 0$

(4)

Compute $\mathcal{L}^{-1}\{F\}$

By definition: $\mathcal{L}^{-1}\{F(w)\} = \int_{-\infty}^{\infty} F(w) e^{-iwx} dw$

$$= \int_{-\infty}^{\infty} e^{-|w|a} e^{-iwx} dw$$

$$= \int_{-\infty}^0 e^{aw} e^{-iwx} dw + \int_0^{\infty} e^{-aw} e^{-iwx} dw$$

$$= \int_{-\infty}^0 e^{w(\alpha - ix)} dw + \int_0^{\infty} e^{w(-\alpha - ix)} dw$$

$$= \frac{1}{\alpha - ix} e^{w(\alpha - ix)} \Big|_{-\infty}^0 + \frac{1}{-\alpha - ix} e^{w(-\alpha - ix)} \Big|_0^{\infty}$$

$$= \frac{1}{\alpha - ix} \left[e^0 - \underbrace{\lim_{b \rightarrow \infty} e^{-b\alpha} e^{ibx}}_{0, \text{ since } \alpha > 0} \right] + \frac{1}{-\alpha - ix} \left[\lim_{b \rightarrow \infty} e^{-b\alpha} e^{-ibx} - e^0 \right]$$

$$= \frac{1}{\alpha - ix} (1) - \frac{1}{\alpha + ix} (-1) = \frac{1}{\alpha - ix} + \frac{1}{\alpha + ix}$$

$$= \frac{2\alpha}{(\alpha - ix)(\alpha + ix)} = \frac{2\alpha}{\alpha^2 + x^2}$$