MATH 3150, Hw #8 solutions

Section 4.2, #1, 5

Section 4.4, # 1

4.2.1 (a) Equation (7) with $Q = -g$:

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2} - g\rho_0(x), \qquad u(0)=u(L)=0$$

Equilibrium: $\frac{\partial^2 u}{\partial t^2} = 0$, $u(x,t) \rightsquigarrow u_E(x)$

$$0 = T_0\frac{\partial^2 u_E}{\partial x^2} - g\rho_0(x)$$

$$u_E''(x) = \frac{g}{T_0}\rho_0(x)$$

Let $P(x)$ be the second antiderivative of $\rho_0(x)$, i.e.

$$P(x) = \int_x \int_t \rho_0(s)\,ds\,dt \quad (\text{so } P''(x) = \rho_0(x))$$

Then $u_E(x) = \frac{g}{T_0}P(x) + c_1 + c_2 x$

Boundary conditions: $u_E(0) = u_E(L) = 0$

$$\left.\begin{aligned} \frac{g}{T_0}P(0) + c_1 &= 0 \\ \frac{g}{T_0}P(L) + c_1 + c_2 L &= 0 \end{aligned}\right\} \Longrightarrow \begin{aligned} c_1 &= -\frac{g}{T_0}P(0) \\ c_2 &= \frac{-g}{LT_0}[P(L) - P(0)] \end{aligned}$$

$$u_E(x) = \frac{g}{T_0}\left[P(x) - P(0) - \frac{x}{L}(P(L) - P(0))\right]$$

(b) Show $v(x,t) = u(x,t) - u_E(x)$ solves $v_{tt} = \frac{T_0}{\rho_0(x)} v_{xx}$

$$v_{tt} = (u - u_E)_{tt} = u_{tt}$$

$$\frac{T_0}{\rho_0(x)} v_{xx} = \frac{T_0}{\rho_0(x)} \left[u_{xx} - (u_E)_{xx} \right]$$

$$= \frac{T_0}{\rho_0(x)} \left[u_{xx} - \frac{g}{T_0} \rho_0(x) \right] = \frac{T_0}{\rho_0(x)} u_{xx} - g$$

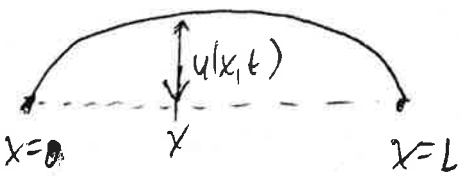
We know: $\rho_0(x) u_{tt} = T_0 u_{xx} - g \rho_0(x)$ (see beginning of part (a))

$$\Downarrow$$

$$\rho_0(x) v_{tt} = \rho_0(x) \left(\frac{T_0}{\rho_0(x)} v_{xx} \right)$$

$$v_{tt} = \frac{T_0}{\rho_0(x)} v_{xx} = c^2 v_{xx} \quad \checkmark$$

#4.2.5

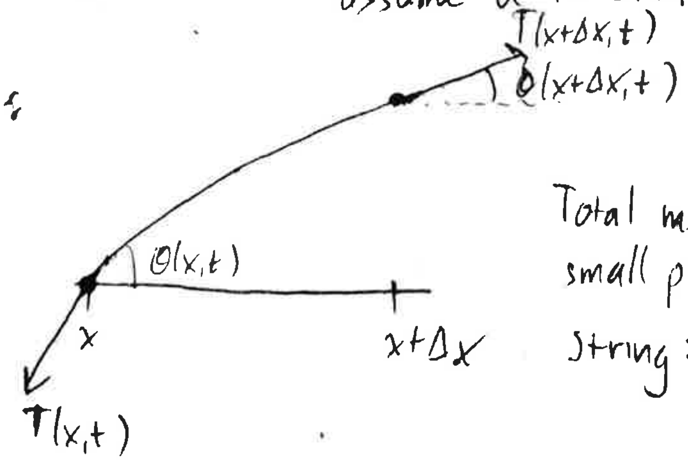


String: constant mass density ρ_0

constant tension T_0.

assume u is small.

Small piece of string:



Total mass of small piece of string $\approx (\Delta x) \rho_0$.

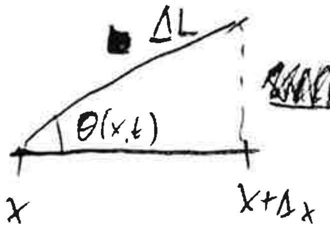
Assume string does not move horizontally $\rightarrow$ forces only act vertically

Newton's 2nd law: $\sum(\text{forces}) = (\text{mass})(\text{acceleration})$

$$[\sin\theta(x+\Delta x,t)\,T(x+\Delta x,t)] - \sin\theta(x,t)\,T(x,t) = (\Delta x \rho_0)\cdot\frac{\partial^2 u}{\partial t^2}$$

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = \frac{T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t)}{\Delta x}$$

$$\approx \frac{\partial}{\partial x}\left[T(x,t)\sin\theta(x,t)\right] \qquad (\text{small } \Delta x)$$



For small displacements, $\Delta L \approx \Delta x$.

$$\sin(\theta(x,t)) \approx \frac{u(x+\Delta x,t) - u(x,t)}{\Delta L}$$

$$\approx \frac{u(x+\Delta x,t) - u(x,t)}{\Delta x}$$

$$\approx \frac{\partial u}{\partial x}$$

so $$\rho_0 \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left[T(x,t)\frac{\partial u}{\partial x}\right]$$

(Whoops.. I forgot that $T(x,t) = T_0$ is constant, but that doesn't affect derivation.)

4.4.1 Constant ρ_0, T_0, define $c^2 = T_0/\rho_0$. (4)

(a) String length L, fixed at endpoints $\Rightarrow$ ~~data~~ $u(0,t) = u(L,t) = 0$.

Wave equation:

$$u_{tt} = c^2 u_{xx}$$

$$u(0,t) = u(L,t) = 0$$

$$u(x,0) = u_0(x)$$

$$\frac{\partial u}{\partial t}(x,0) = v_0(x)$$

some initial data. (we won't really need it)

Separation of variables: $u(x,t) = \varphi(x) h(t)$

$$\Downarrow$$

$$\frac{h''(t)}{c^2 h(t)} = \frac{\varphi''(x)}{\varphi(x)} = -\lambda \text{ (unknown constant)}$$

Boundary conditions: $u(0,t) = 0 \Rightarrow \varphi(0) = 0$
$u(L,t) = 0 \Rightarrow \varphi(L) = 0$

$$\begin{cases} \varphi''(x) + \lambda \varphi(x) \\ \varphi(0) = \varphi(L) = 0 \end{cases}$$

Find λ that allow nontrivial solutions $\varphi(x)$.

$\lambda < 0$: no eigenvalues

$\lambda = 0$: no eigenvalues

$\lambda > 0$: $\varphi(x) = c_1 \cos(x\sqrt{\lambda}) + c_2 \sin(x\sqrt{\lambda})$

$\varphi(0) = 0 \Rightarrow c_1 = 0$

$\varphi(L) = 0 \Rightarrow c_2 \sin(L\sqrt{\lambda}) = 0 \Rightarrow \lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n = 1, 2, \ldots$

$\Phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$

Function $h_n(t)$ associated to each λ_n:

$$\frac{h_n''(t)}{c^2 h_n(t)} = -\lambda_n \Rightarrow h_n''(t) + \lambda_n c^2 h_n(t) = 0$$

$$h_n(t) = a_n \cos\left(t\sqrt{c^2\lambda_n}\right) + b_n \sin\left(t\sqrt{c^2\lambda_n}\right)$$

$u_n(x,t) = \varphi_n(x)\, h_n(t)$

Superposition: $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$

$$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[a_n \cos\left(\frac{n\pi c t}{L}\right) + b_n \sin\left(\frac{n\pi c t}{L}\right)\right]$$

initial data: $u(x,0) = u_0(x)$ $\qquad \frac{\partial u}{\partial t}(x,0) = v_0(x)$

$$\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) = u_0(x) \qquad \sum_{n=1}^{\infty} b_n \frac{n\pi c}{L} \sin\left(\frac{n\pi x}{L}\right) = v_0(x)$$

Fourier sine series (or orthogonality)

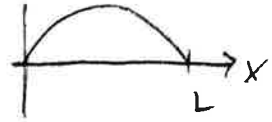
$$\Rightarrow a_n = \frac{2}{L}\int_0^L u_0(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{2}{n\pi c}\int_0^L v_0(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

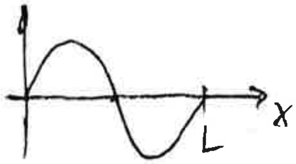
$$u(x,t) = \sum_{n=1}^{\infty} \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\text{normal modes}}\left[a_n \cos\left(\frac{n\pi c t}{L}\right) + b_n \sin\left(\frac{n\pi c t}{L}\right)\right]$$

$\frac{n\pi c}{L}$: natural frequencies.

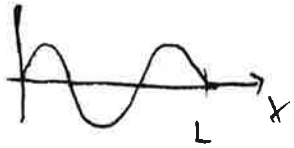
Natural frequencies: $\frac{n\pi c}{L}$, $n = 1, 2, \ldots$



$n=1$



$n=2$ normal modes.



$n=3$

(b) $\underbrace{\text{String fixed at } x=0}_{u(0,t)=0}$, $\underbrace{\text{"free" at } x=H}_{\frac{\partial u}{\partial x}(H,t)=0}$

$$u_{tt}=c^2u_{xx}$$
$$u(0,t)=0$$
$$\frac{\partial u}{\partial x}(H,t)=0$$
$$u(x,0)=u_0(x)$$
$$\frac{\partial u}{\partial t}(x,0)=v_0(x)$$

some initial data.

Separation of variables: $u(x,t)=\phi(x)\,h(t)$

$$\Downarrow$$

$$\frac{h''(t)}{c^2h(t)}=\frac{\phi''(x)}{\phi(x)}=-\lambda \text{ (unknown constant)}$$

$$u(0,t)=0 \Rightarrow \phi(0)=0$$

$$\frac{\partial u}{\partial x}(H,t)=0 \Rightarrow \phi'(H)=0$$

$$\begin{cases} \varphi''(x) + \lambda \varphi(x) = 0 \\ \varphi(0) = 0, \quad \varphi'(H) = 0 \end{cases}$$

$\underline{\lambda < 0}$: no eigenvalues

$\underline{\lambda = 0}$: no eigenvalues

$\underline{\lambda > 0}$: $\varphi(x) = c_1 \cos(x\sqrt{\lambda}) + c_2 \sin(x\sqrt{\lambda})$

$$\varphi(0) = 0 \Rightarrow c_1 = 0$$

$$\varphi'(H) = 0 \Rightarrow \sqrt{\lambda}\, c_2 \cos(H\sqrt{\lambda}) = 0 \Rightarrow \lambda = \lambda_n = \left(\frac{(2n-1)\pi}{2H}\right)^2$$

$n = 1, 2 \ldots$

$$\varphi_n(x) = \sin(x\sqrt{\lambda_n}) = \sin\left(\frac{(2n-1)\pi x}{2H}\right)$$

$$h_n''(t) + c^2 \lambda_n h_n(t) = 0$$

$$h_n(t) = a_n \cos(tc\sqrt{\lambda_n}) + b_n \sin(tc\sqrt{\lambda_n})$$

$u_n(x,t) = \varphi_n(x)\, h_n(t)$

Full solution: $u(x,t) = \sum_{n=1}^{\infty} \varphi_n(x)\, h_n(t)$

$$= \sum_{n=1}^{\infty} \sin\left(\frac{(2n-1)\pi x}{2H}\right) \cdot \left[a_n \cos\left(\frac{(2n-1)\pi c t}{2H}\right) + b_n \sin\left(\frac{(2n-1)\pi t c}{2H}\right)\right]$$

initial data: $u(x,0) = u_0(x)$ $\qquad$ $\frac{\partial u}{\partial t}(x,0) = v_0(x)$

$\downarrow$ $\qquad$ $\downarrow$

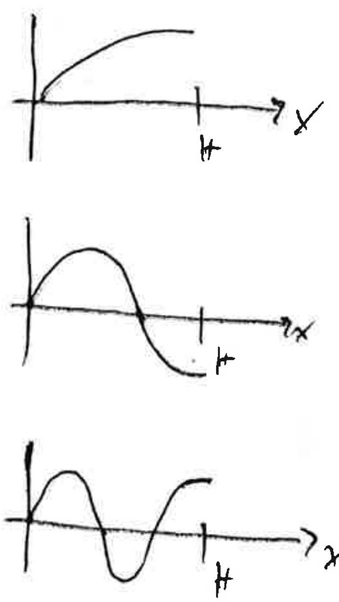
$$\sum_{n=1}^{\infty} a_n \sin\left(\frac{(2n-1)\pi x}{2H}\right) = u_0(x) \qquad \sum_{n=1}^{\infty} b_n \cdot \frac{(2n-1)\pi c}{2H} \sin\left(\frac{(2n-1)\pi x}{2H}\right) = v_0(x)$$

Orthogonality $\Longrightarrow$ $a_n = \frac{2}{H}\int_0^H u_0(x)\sin\left(\frac{(2n-1)\pi x}{2H}\right)dx$

$$b_n = \frac{4}{(2n-1)\pi c}\int_0^H v_0(x)\sin\left(\frac{(2n-1)\pi x}{2H}\right)dx$$

$$u(x,t) = \sum_{n=1}^{\infty} \underbrace{\sin\left(\frac{(2n-1)\pi x}{2H}\right)}_{\text{normal modes}} \left[a_n\cos\left(\frac{(2n-1)\pi c t}{2H}\right) + b_n \sin\left(\frac{(2n-1)\pi c t}{2H}\right)\right].$$

natural frequencies: $\frac{(2n-1)\pi c}{2H}$, $n=1,2,\ldots$



$n=1$

$n=2$

$n=3$

normal modes

(c) $H = L/2$ odd modes from (a): $\sin\left(\frac{\pi x}{L}\right)$

$\sin\left(\frac{3\pi x}{L}\right)$

$\sin\left(\frac{5\pi x}{L}\right)$

$\vdots$

modes from (b): $\sin\left(\frac{\pi x}{2H}\right) \xrightarrow{H=L/2} \sin\left(\frac{\pi x}{L}\right)$ ✓

$\sin\left(\frac{3\pi x}{2H}\right) \xrightarrow{H=L/2} \sin\left(\frac{3\pi x}{L}\right)$ ✓

$\sin\left(\frac{5\pi x}{2H}\right) \xrightarrow{H=L/2} \sin\left(5\pi x/L\right)$ ✓

More generally: $\sin\left(\frac{(2n-1)\pi x}{L}\right) = \sin\left(\frac{(2n-1)\pi x}{2H}\right)$ since $H = L/2$

odd modes from (a) — modes from (b)

Odd natural frequencies from (a): $\frac{(2n-1)\pi c}{L}$

natural frequencies from (b): $\frac{(2n-1)\pi c}{2H}$

They match, using $H = L/2$

One can see from the plots of the normal modes that extending the part (b) modes up to $x = 2H$ makes them identical to the part (a) odd modes. (I.e. the odd modes from (a) are symmetric around $x = L/2$, and match the modes from (b)).