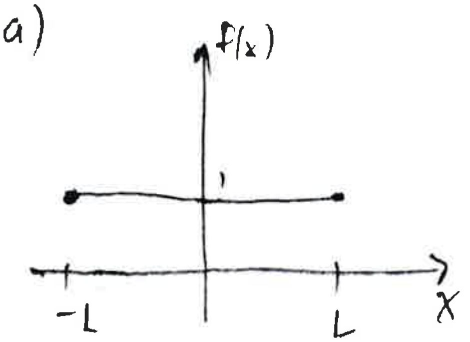
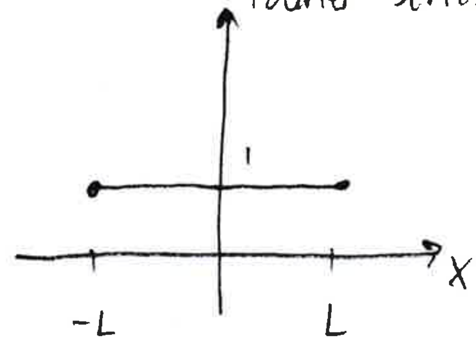
MATH 3150 HW #6 solutions

3.2.1 (a)

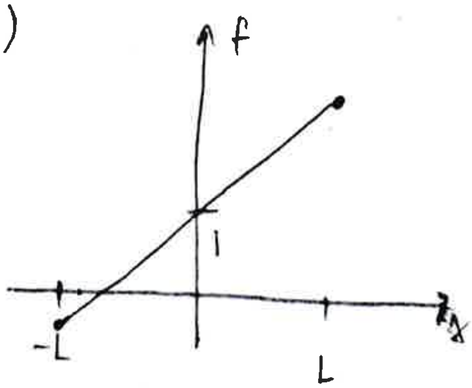
Fourier Series (FS)

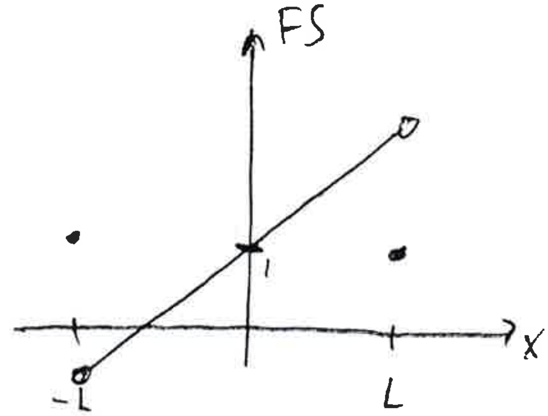




f(x) and its Fourier Series are equal.

(c)

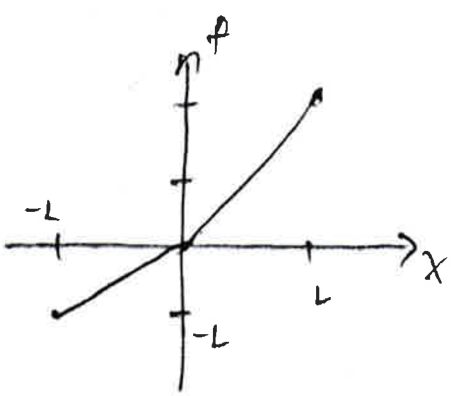


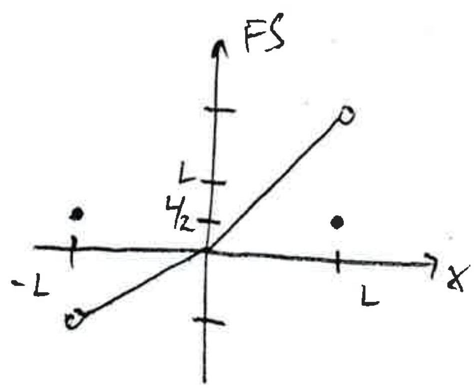


(As pictured, $L > 1$)

f and its Fourier Series are equal on the interior $(-L, L)$, but differ at $x = \pm L$.

(e)

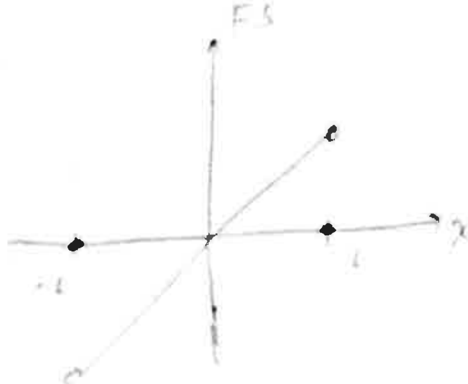




f and its Fourier Series are equal on the interior $(-L, L)$, but differ at $x = \pm L$.

②

3.2.2 (a) $f(x) = x$



FS: $\sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$

$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$

$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad (n \neq 0)$

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

$a_0 = \frac{1}{2L}\int_{-L}^{L} x\,dx = 0$

$a_n = \frac{1}{L}\int_{-L}^{L} x\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\left[\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx + \int_{-L}^{0} x\cos\left(\frac{n\pi x}{L}\right)dx\right]$

$= \frac{1}{L}\left[\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx - \int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx\right]$

$= 0$

$b_n = \frac{1}{L}\int_{-L}^{L} x\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)dx$

$u = x \qquad v = -\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)$

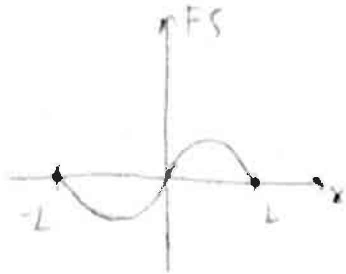
$du = dx \qquad dv = \sin\left(\frac{n\pi x}{L}\right)dx$

$= \frac{2}{L}\left[-\frac{L}{n\pi}x\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L + \frac{L}{n\pi}\int_0^L \cos\left(\frac{n\pi x}{L}\right)dx\right]$

$= \frac{2}{L}\left[-\frac{L^2}{n\pi}\cos(n\pi) + \frac{L^2}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L\right]$

$= \frac{2L}{n\pi}(-1)^{n+1}$

(c) $f(x) = \sin\left(\frac{\pi x}{L}\right)$



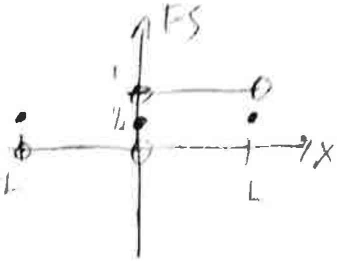
$a_0 = \frac{1}{2L}\int_{-L}^{L}\sin\left(\frac{\pi x}{L}\right)dx = 0$

$a_n = \frac{1}{L}\int_{-L}^{L}\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = 0$

(see midterm 1 formula sheet)

$b_n = \frac{1}{L}\int_{-L}^{L}\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \begin{cases} 0, & n \neq 1 \\ 1, & n = 1 \end{cases}$

(f) $f(x) = \begin{cases} 0, & x<0 \\ 1, & x>0 \end{cases}$



$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_0^L dx = \frac{1}{2}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_0^L \cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx = -\frac{1}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L$$

$$= -\frac{1}{n\pi}\left[(-1)^n - 1\right]$$

$$= \frac{1}{n\pi}\left[1-(-1)^{n+1}\right]$$

3.2.3 FS of $f_1(x)$ is $\sum_{n=0}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$

FS of $f_2(x)$ is $\sum_{n=0}^{\infty} e_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} d_n \sin\left(\frac{n\pi x}{L}\right)$

FS of $c_1f_1(x)+c_2f_2(x)$ is $\sum_{n=0}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$

where: $a_n = \begin{cases} \frac{1}{2L}\int_{-L}^{L} f_1(x)\,dx, & n=0 \\ \frac{1}{L}\int_{-L}^{L} f_1(x)\cos\left(\frac{n\pi x}{L}\right)dx, & n>0 \end{cases}$ $\quad b_n = \frac{1}{L}\int_{-L}^{L} f_1(x)\sin\left(\frac{n\pi x}{L}\right)dx$

$e_n = \begin{cases} \frac{1}{2L}\int_{-L}^{L} f_2(x)\,dx, & n=0 \\ \frac{1}{L}\int_{-L}^{L} f_2(x)\cos\left(\frac{n\pi x}{L}\right)dx, & n>0 \end{cases}$ $\quad d_n = \frac{1}{L}\int_{-L}^{L} f_2(x)\sin\left(\frac{n\pi x}{L}\right)dx$

$A_n = \begin{cases} \frac{1}{2L}\int_{-L}^{L} \left(c_1f_1(x)+c_2f_2(x)\right)dx, & n=0 \\ \frac{1}{L}\int_{-L}^{L} \left(c_1f_1(x)+c_2f_2(x)\right)\cos\left(\frac{n\pi x}{L}\right)dx, & n>0 \end{cases}$

$B_n = \frac{1}{L}\int_{-L}^{L}\left(c_1f_1(x)+c_2f_2(x)\right)\sin\left(\frac{n\pi x}{L}\right)dx$

By linearity of integral: $A_n = c_1a_n + c_2e_n$, $B_n = c_1b_n + c_2d_n$

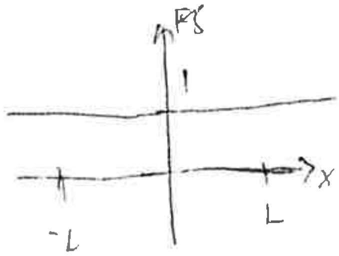
Thus,

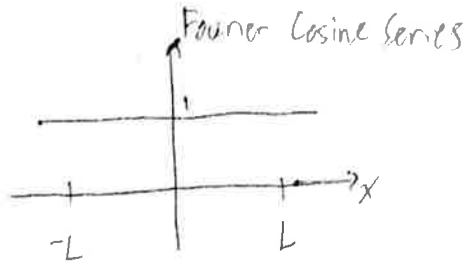
$$c_1(\text{FS of } f_1) + c_2(\text{FS of } f_2)$$

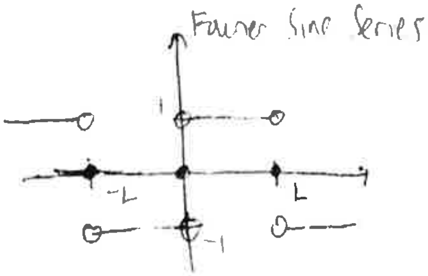
$$= c_1\left[\sum_{n=0}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right)\right] + c_2\left[\sum_{n=0}^{\infty} e_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} d_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

$$= \sum_{n=0}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi x}{L}\right) = (\text{FS of } c_1f_1(x) + c_2f_2(x))$$

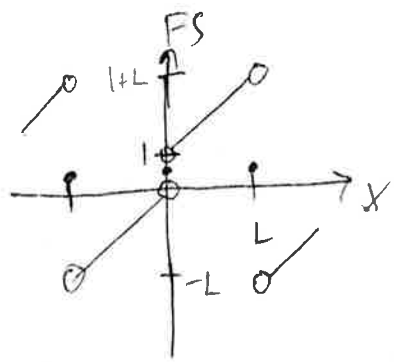
3.3.1 (a) $f(x) = 1$

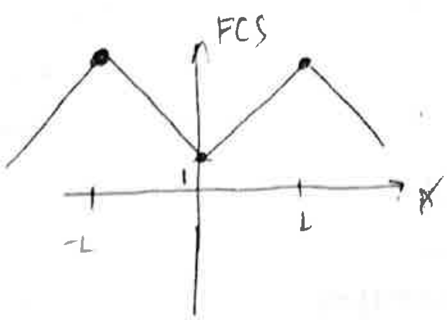


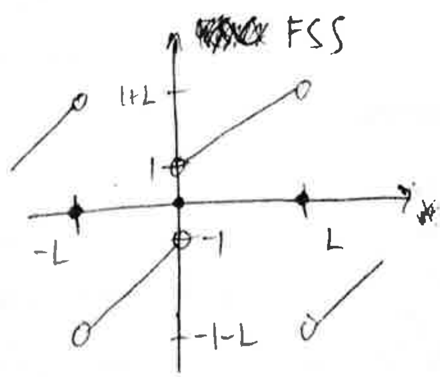




(c) $f(x) = \begin{cases} x, & x < 0 \\ 1+x, & x > 0 \end{cases}$







3.3.2 (a) $f(x) = \cos\left(\frac{\pi x}{L}\right)$

Fourier sine series coefficients: $B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$, $n=1,2$

Fourier sine series: $f_{FSS}(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$

$$B_n = \frac{2}{L}\int_0^L \cos\left(\frac{\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

Since $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$, then

$$\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right) = \left[\sin\left(\frac{(n+1)\pi x}{L}\right) + \sin\left(\frac{(n-1)\pi x}{L}\right)\right]\cdot\frac{1}{2}$$

So: $$B_n = \frac{1}{L}\int_0^L \sin\left(\frac{(n+1)\pi x}{L}\right)dx + \frac{1}{L}\int_0^L \sin\left(\frac{(n-1)\pi x}{L}\right)dx$$

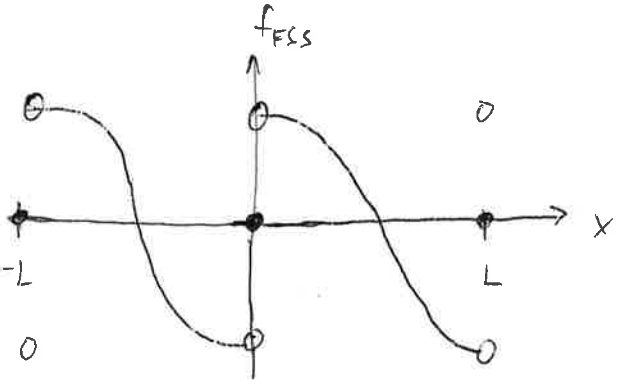
$$= \frac{-1}{(n+1)\pi}\cos\left(\frac{(n+1)\pi x}{L}\right)\Big|_0^L - \frac{1}{(n-1)\pi}\cos\left(\frac{(n-1)\pi x}{L}\right)\Big|_0^L$$

$$= \frac{-1}{\pi(n+1)}\left[\cos((n+1)\pi) - 1\right] - \frac{1}{(n-1)\pi}\left[\cos((n-1)\pi) - 1\right]$$

if n is odd: $\cos((n\pm 1)\pi) = 1 \Rightarrow B_n = 0$

if n is even: $\cos((n\pm 1)\pi) = -1 \Rightarrow B_n = \frac{2}{\pi(n+1)} + \frac{2}{\pi(n-1)} = \frac{4n}{\pi(n^2-1)}$

$$B_n = \begin{cases} 0, & n \text{ odd} \\ \frac{4n}{\pi(n^2-1)}, & n \text{ even} \end{cases}$$ (matches (3.3.13)).



(c) $f(x) = \begin{cases} 0, & x < L/2 \\ x, & x > L/2 \end{cases}$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_{L/2}^{L} x\sin\left(\frac{n\pi x}{L}\right)dx$$

$$u = x \qquad v = \frac{-L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)$$

$$du = dx \qquad dv = \sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\left[\frac{-L}{n\pi}x\cos\left(\frac{n\pi x}{L}\right)\Big|_{L/2}^{L} + \frac{L}{n\pi}\int_{L/2}^{L}\cos\left(\frac{n\pi x}{L}\right)dx\right]$$

$$= \frac{-2}{n\pi}\left[L\cos(n\pi) - \frac{L}{2}\cos\left(\frac{n\pi}{2}\right)\right]$$

$$+ \frac{2}{n\pi}\cdot\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{L/2}^{L}$$

$$= \frac{-2}{n\pi}\left[(-1)^n L - \frac{L}{2}\cos\left(\frac{n\pi}{2}\right)\right] + \frac{2L}{(n\pi)^2}\left[\cancel{\sin(n\pi)}^{\,0} - \sin\left(\frac{n\pi}{2}\right)\right]$$

$$B_n = \frac{(-1)^{n+1}2L}{n\pi} + \frac{L}{n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{2L}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)$$