MATH 3150 HW # 6

Section 2.5 # 1(a, c, d), 28

2.5.1 (a) $u_{xx} + u_{yy} = 0$, $0 < x < L$, $0 < y < H$

$$\frac{\partial u}{\partial x}(0,y) = \frac{\partial u}{\partial x}(L,y) = 0$$

$$u(x,0) = 0 \qquad u(x,H) = f(x)$$

Separation of variables: $u(x,y) = \varphi(x) g(y)$

$$u_{xx} + u_{yy} = 0 \Longrightarrow \frac{\varphi''(x)}{\varphi(x)} = \frac{-g''(y)}{g(y)} = -\lambda$$

unknown constant

$$\frac{\partial u}{\partial x}(0,y) = \frac{\partial u}{\partial x}(L,y) = 0 \Longrightarrow \varphi'(0) = \varphi'(L) = 0$$

$$u(x,0) = 0 \Longrightarrow g(0) = 0$$

Since φ has 2 homogeneous boundary conditions, we start with that:

$$\begin{cases} \varphi''(x) + \lambda\varphi(x) = 0 \\ \varphi'(0) = \varphi'(L) = 0 \end{cases}$$

$\lambda < 0 \Longrightarrow \varphi(x) = c_1 \cosh(x\sqrt{|\lambda|}) + c_2 \sinh(x\sqrt{|\lambda|})$.

$$\varphi'(x) = c_1\sqrt{|\lambda|}\sinh(x\sqrt{|\lambda|}) + c_2\sqrt{|\lambda|}\cosh(x\sqrt{|\lambda|})$$

$$\varphi'(0) = 0 \Longrightarrow c_2 = 0$$

$$\varphi'(L) = 0 \Longrightarrow c_1\sqrt{|\lambda|}\sinh(L\sqrt{|\lambda|}) = 0 \Rightarrow c_1 = 0$$

No negative eigenvalues

$\lambda = 0 \Longrightarrow \varphi(x) = c_1 + c_2 x$

$$\varphi'(x) = c_2$$

$$\varphi'(0) = 0 \Longrightarrow c_2 = 0, \quad \varphi'(L) = 0 \Longrightarrow c_2 = 0$$

c_1 arbitrary

$\lambda = 0$ is an eigenvalue with eigenfunction $\varphi_0(x) = 1$.

$\lambda > 0 \Rightarrow \varphi(x) = c_1 \cos(x\sqrt{\lambda}) + c_2 \sin(x\sqrt{\lambda})$

$\varphi'(x) = -c_1\sqrt{\lambda}\sin(x\sqrt{\lambda}) + c_2\sqrt{\lambda}\cos(x\sqrt{\lambda})$

$\varphi'(0) = 0 \Rightarrow c_2 = 0$

$\varphi'(L) = 0 \Rightarrow -c_1\sqrt{\lambda}\sin(L\sqrt{\lambda}) = 0$

$L\sqrt{\lambda} = n\pi, \quad n = 1, 2 ..$

$\lambda_n = (\frac{n\pi}{L})^2$ are eigenvalues with eigenfunctions $\varphi_n(x) = \cos(x\sqrt{\lambda_n})$

$= \cos(\frac{n\pi x}{L})$

Fix n. With $\lambda = \lambda_n$:

$g_n'' - \lambda_n g_n = 0$

$g_n(0) = 0$

$\underline{n=0}$: $g_0'' = 0 \Rightarrow g_0(y) = c_1 + c_2 y$

$g_0(0) = 0 \Rightarrow c_1 = 0$

$g_0(y) = y$ (choose $c_2 = 1$)

$\underline{n>0}$: $g_n'' - (\frac{n\pi}{L})^2 g_n = 0$

$g_n(y) = c_1\cosh(^{n\pi y}/_L) + c_2\sinh(^{n\pi y}/_L)$

$g_n(0) = 0 \Rightarrow c_1 = 0$

$g_n(y) = \sinh(^{n\pi y}/_L)$.

Superposition: $u(x,y) = \sum_{n=0}^{\infty} a_n \varphi_n(x) g_n(y)$.

$= a_0 y + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L})\sinh(\frac{n\pi y}{L})$.

Last BC: $u(x,H) = f(x)$.

$f(x) \overset{?}{=} a_0 H + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L})\sinh(\frac{n\pi H}{L})$

Orthogonality: $\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0, & m\neq n \\ L, & m=n=0 \\ L/2, & m=n\neq 0 \end{cases}$

multiply by $\cos(m\pi x/L)$, integrate from 0 to L:

$$\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)dx = \frac{L}{2}a_m\sinh\left(\frac{m\pi H}{L}\right) \quad (m>0)$$

$$\int_0^L f(x)\,dx = La_0H \quad (m=0).$$

$$\Rightarrow \boxed{\begin{aligned} a_0 &= \frac{1}{LH}\int_0^L f(x)\,dx \\ a_m &= \frac{2}{L\sinh\left(\frac{m\pi H}{L}\right)}\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)dx \quad (m>0) \\ u(x,y) &= a_0y+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right) \end{aligned}}$$

2.5.1 (c) $u_{xx} + u_{yy} = 0, \quad 0 < x < L, \; 0 < y < H$

$$\frac{\partial u}{\partial x}(0,y) = 0 \qquad u(L,y) = g(y)$$

$$u(x,0) = 0 \qquad u(x,H) = 0$$

Separation of variables: $u(x,y) = g(x)\,\varphi(y)$.

$$u_{xx} + u_{yy} = 0 \Rightarrow \frac{\varphi''(y)}{\varphi(y)} = \frac{-g''(x)}{g(x)} = -\lambda \text{ (unknown constant)}$$

$$\frac{\partial u}{\partial x}(0,y) = 0 \Rightarrow g'(0) = 0$$

$$u(x,0) = 0 \Rightarrow \varphi(0) = 0$$

$$u(x,H) = 0 \Rightarrow \varphi(H) = 0$$

Start with equation having 2 homogeneous boundary conditions:

$$\begin{cases} \varphi''(y) + \lambda\varphi(y) = 0 \\ \varphi(0) = \varphi(H) = 0 \end{cases}$$

$\lambda < 0 \Rightarrow \varphi(x) = c_1 \cosh(y\sqrt{|\lambda|}) + c_2 \sinh(y\sqrt{|\lambda|})$.

$\varphi(0) = 0 \Rightarrow c_1 = 0$

$\varphi(H) = 0 \Rightarrow c_2 \sinh(H\sqrt{|\lambda|}) = 0 \Rightarrow c_2 = 0 \Rightarrow$ no negative eigenvalues

$\lambda = 0 \Rightarrow \varphi(y) = c_1 + c_2 y$

$\varphi(0) = 0 \Rightarrow c_1 = 0$

$\varphi(H) = 0 \Rightarrow c_2 = 0$

$\lambda = 0$ is not an eigenvalue.

$\lambda > 0 \Rightarrow \varphi(y) = c_1 \cos(y\sqrt{\lambda}) + c_2 \sin(y\sqrt{\lambda})$

$\varphi(0) = 0 \Rightarrow c_1 = 0$

$\varphi(H) = 0 \Rightarrow c_2 \sin(H\sqrt{\lambda}) = 0 \Rightarrow \lambda = \lambda_n = \left(\frac{n\pi}{H}\right)^2, \; n = 1, 2, \ldots$

Eigenvalues: $\lambda_n = \left(\frac{n\pi}{H}\right)^2$

Eigenfunctions: $\Phi_n(y) = \sin\left(\frac{n\pi y}{H}\right)$, $n = 1, 2, \ldots$

For fixed n:

$$q_n''(x) - \lambda_n q_n(x) = 0$$
$$q_n'(0) = 0$$

$$q_n(x) = c_1 \cosh(x\sqrt{\lambda_n}) + c_2 \sinh(x\sqrt{\lambda_n})$$
$$q_n'(0) = 0 \Rightarrow c_2 = 0$$
$$q_n(x) = c_n \cosh(x\sqrt{\lambda_n})$$

Superposition: $u(x,y) = \sum_{n=1}^{\infty} b_n \cosh(x\sqrt{\lambda_n}) \sin\left(\frac{n\pi y}{H}\right)$

Remaining boundary condition: $u(L,y) = g(y)$

$$\sum_{n=1}^{\infty} b_n \cosh(L\sqrt{\lambda_n}) \sin\left(\frac{n\pi y}{H}\right) = g(y)$$

Orthogonality: multiply both sides by $\sin\left(\frac{m\pi y}{H}\right)$ and integrate:

$$\sum_{n=1}^{\infty} b_n \cosh(L\sqrt{\lambda_n}) \int_0^H \sin\left(\frac{n\pi y}{H}\right) \sin\left(\frac{m\pi y}{H}\right) dy = \int_0^H g(y) \sin\left(\frac{m\pi y}{H}\right) dy$$

$$b_m \cosh(L\sqrt{\lambda_m}) \cdot \frac{H}{2} = \int_0^H g(y) \sin\left(\frac{m\pi y}{H}\right) dy$$

$$\boxed{\begin{gathered} b_m = \frac{2}{H\cosh(L\sqrt{\lambda_m})} \int_0^H g(y) \sin\left(\frac{m\pi y}{H}\right) dy \\ u(x,y) = \sum_{n=1}^{\infty} b_n \cosh(x\sqrt{\lambda_n}) \sin(y\sqrt{\lambda_n}) \\ \lambda_n = \left(\frac{n\pi}{H}\right)^2, \quad n = 1, 2 \ldots \end{gathered}}$$

2.5.1 (d) $u_{xx} + u_{yy} = 0$

$u(0,y) = g(y) \quad u(L,y) = 0$

$\frac{\partial u}{\partial y}(x,0) = 0, \quad u(x,H) = 0.$

Separation of variables: $u(x,y) = g(x)\varphi(y)$.

PDE $\Longrightarrow \frac{\varphi''(y)}{\varphi(y)} = -\frac{g''(x)}{g(x)} = -\lambda$ (unknown constant)

$u(L,y) = 0 \Rightarrow g(L) = 0$

$\frac{\partial u}{\partial y}(x,0) = 0 \Rightarrow \varphi'(0) = 0$

$u(x,H) = 0 \Longrightarrow \varphi(H) = 0$

Start with function where we have 2 homogeneous boundary conditions

$$\begin{cases} \varphi'' + \lambda\varphi = 0 \\ \varphi'(0) = 0, \quad \varphi(H) = 0 \end{cases}$$

$\lambda < 0 \Longrightarrow \varphi(y) = c_1 \cosh(y\sqrt{|\lambda|}) + c_2 \sinh(y\sqrt{|\lambda|})$

$\varphi'(0) = 0 \Longrightarrow c_2 = 0$

$\varphi(H) = 0 \Longrightarrow c_1 \cosh(H\sqrt{|\lambda|}) = 0 \Rightarrow c_1 = 0$

Trivial solution, so no negative eigenvalues

$\lambda = 0 \Longrightarrow \varphi(y) = c_1 + c_2 y$

$\varphi'(0) = 0 \Rightarrow c_2 = 0$

$\varphi(H) = 0 \Rightarrow c_1 = 0$

$\lambda = 0$ is not an eigenvalue

$\lambda > 0 \Longrightarrow \varphi(y) = c_1 \cos(y\sqrt{\lambda}) + c_2 \sin(y\sqrt{\lambda})$

$\varphi'(0) = 0 \Rightarrow c_2 = 0$

$\varphi(H) = 0 \Longrightarrow c_1 \cos(H\sqrt{\lambda}) = 0 \Rightarrow \lambda = \lambda_n = \left(\frac{(2n-1)\pi}{2H}\right)^2, \; n = 1, 2 \ldots$

Eigenvalues: $\lambda_n = \left(\frac{(2n-1)\pi}{2H}\right)^2, \quad n=1,2,\ldots$

Eigenfunctions: $\varphi_n(y) = \cos(y\sqrt{\lambda_n})$

For each fixed $n=1,2\ldots$:

$$g_n''(x) - \lambda_n g_n(x) = 0$$
$$g(L) = 0$$

$$g_n(x) = c_1\cosh(x\sqrt{\lambda_n}) + c_2\sinh(x\sqrt{\lambda_n}) \qquad (\text{or } c_1\cosh((x-L)\sqrt{\lambda_n}) + c_2\sinh((x-L)\sqrt{\lambda_n}))$$

$$g(L) = 0 \Rightarrow c_1\cosh(L\sqrt{\lambda_n}) + c_2\sinh(L\sqrt{\lambda_n}) = 0$$
$$c_1 = -c_2\tanh(L\sqrt{\lambda_n})$$

$$g_n(x) = c_2\left[\sinh(x\sqrt{\lambda_n}) - \tanh(L\sqrt{\lambda_n})\cosh(x\sqrt{\lambda_n})\right] \quad (\text{chose } c_2 = 1).$$

Superposition: $u(x,y) = \sum_{n=1}^{\infty} a_n \varphi_n(y)\, g_n(x)$

Last boundary condition: $u(0,y) = g(y)$

$$\sum_{n=1}^{\infty} a_n \varphi_n(y)\, g_n(0) = g(y)$$

Multiply by $\varphi_m(y)$, integrate from 0 to H:

$$\sum_{n=1}^{\infty} a_n g_n(0) \int_0^H \varphi_n(y)\varphi_m(y)\,dy = \int_0^H \varphi_m(y) g(y)\,dy$$

$$a_m g_m(0)\, H/2 = \int_0^H \varphi_m(y) g(y)\,dy$$

$$\boxed{\begin{aligned} & a_m = \frac{2}{H g_m(0)} \int_0^H \varphi_m(y) g(y)\,dy, \quad m=1,2\ldots \\ & \varphi_n(y) = \cos(y\sqrt{\lambda_n}), \quad n=1,2\ldots \\ & g_n(x) = \sinh(x\sqrt{\lambda_n}) - \tanh(L\sqrt{\lambda_n})\cos(x\sqrt{\lambda_n}) \\ & \lambda_n = \left(\frac{(2n-1)\pi}{2H}\right)^2 \\ & u(x,y) = \sum_{n=1}^{\infty} a_n \varphi_n(y)\, g_n(x) \end{aligned}}$$

2.5.28

$$u_{xx}+u_{yy}=0$$

$$u(0,y)=g(y) \qquad u(x,0)=0$$

$$u(L,y)=0 \qquad u(x,H)=0$$

Separation of variables: $u(x,y)=\varphi(y)\,g(x)$

$$u_{xx}+u_{yy}=0 \implies \frac{\varphi''(y)}{\varphi(y)}=-\frac{g''(x)}{g(x)}=-\lambda \quad \text{(unknown constant)}.$$

$$u(x,0)=0 \implies \varphi(0)=0$$

$$u(L,y)=0 \implies g(L)=0$$

$$u(x,H)=0 \implies \varphi(H)=0.$$

Start with ODE for φ:

$$\begin{cases} \varphi''(y)+\lambda\varphi(y)=0 \\ \varphi(0)=0,\ \varphi(H)=0 \end{cases}$$

$\lambda<0 \implies \varphi(y)=c_1\cosh(y\sqrt{|\lambda|})+c_2\sinh(y\sqrt{|\lambda|})$

$\varphi(0)=0 \implies c_1=0$

$\varphi(H)=0 \implies c_2\sinh(H\sqrt{|\lambda|})=0 \implies c_2=0$

Trivial solution, so $\lambda<0$ is not an eigenvalue.

$\lambda=0 \implies \varphi(y)=c_1+c_2y$

$\varphi(0)=0 \implies c_1=0$

$\varphi(H)=0 \implies c_2=0$

Trivial solution, so $\lambda=0$ is not an eigenvalue

$\lambda<0 \implies \varphi(y)=c_1\cos(y\sqrt{\lambda})+c_2\sin(y\sqrt{\lambda})$

$\varphi(0)=0 \implies c_1=0$

$\varphi(H)=0 \implies c_2\sin(H\sqrt{\lambda})=0 \implies \lambda=\lambda_n=\left(\frac{n\pi}{H}\right)^2,\ n=1,2,3,\ldots$

Eigenvalues: $\lambda_n = \left(\frac{n\pi}{H}\right)^2$, $n=1,2,3,\ldots$

Eigenfunctions: $\varphi_n(y) = \sin(y\sqrt{\lambda_n})$

For each fixed $n=1,2\ldots$:

$$g_n''(x) - \lambda_n g_n(x) = 0$$

$$g_n(L) = 0$$

$$g_n(x) = c_1 \cosh((x-L)\sqrt{\lambda_n}) + c_2 \sinh((x-L)\sqrt{\lambda_n}) \quad (\text{or } c_1 \cosh(x\sqrt{\lambda_n}) + c_2 \sinh(x\sqrt{\lambda_n}))$$

$$g_n(L) = 0 \Rightarrow c_1 = 0$$

$$g_n(x) = c_2 \sinh((x-L)\sqrt{\lambda_n}) \quad (\text{choose } c_2 = 1)$$

Superposition: $u(x,y) = \sum_{n=1}^{\infty} b_n \varphi_n(y) g_n(x)$

Last boundary condition: $u(0,y) = g(y)$

$$\sum_{n=1}^{\infty} b_n \varphi_n(y) g_n(0) = g(y)$$

Multiply both sides by $\varphi_m(y)$, integrate from 0 to H:

$$\sum_{n=1}^{\infty} b_n g_n(0) \int_0^H \varphi_n(y) \varphi_m(y)\, dy = \int_0^H g(y) \varphi_m(y)\, dy$$

$$\boxed{\begin{aligned} b_m &= \frac{2}{H g_n(0)} \int_0^H g(y) \varphi_m(y)\, dy \\ \varphi_n(y) &= \sin(y\sqrt{\lambda_n}), \quad n=1,2\ldots \\ g_n(x) &= \sinh((x-L)\sqrt{\lambda_n}) \\ \lambda_n &= (n\pi/H)^2 \\ u(x,y) &= \sum_{n=1}^{\infty} b_n g_n(x) \varphi_n(y) \end{aligned}}$$