

2.3.7 $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial u}{\partial x}(0, t) = 0 \quad \frac{\partial u}{\partial x}(L, t) = 0$

$$u(x, 0) = f(x)$$

(a) This problem models temperature distribution in a one-dimensional rod whose endpoints are insulated with initial temperature distribution $f(x)$.

(b) Separation of variables: $u(x, t) = \phi(x)T(t)$

$$u_t = k u_{xx} \longrightarrow T'(t)\phi(x) = k T(t)\phi''(x)$$

$$\frac{T'(t)}{kT(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda \leftarrow \text{unknown constant}$$

$$\frac{\partial u}{\partial x}(0, t) = 0 \Rightarrow \phi'(0) = 0$$

$$\frac{\partial u}{\partial x}(L, t) = 0 \Rightarrow \phi'(L) = 0$$

$$\text{ODE for } \phi: \begin{cases} \phi''(x) + \lambda \phi(x) = 0 \\ \phi'(0) = 0, \quad \phi'(L) = 0 \end{cases}$$

$$\underline{\lambda < 0} \quad \phi''(x) - |\lambda| \phi(x) = 0$$

$$r^2 - |\lambda| = 0 \Rightarrow r = \pm \sqrt{|\lambda|}$$

$$r_1 = \sqrt{|\lambda|}, \quad r_2 = -\sqrt{|\lambda|} \quad \text{and} \quad r_1 \neq r_2$$

$$\phi(x) = c_1 \exp(r_1 x) + c_2 \exp(r_2 x)$$

$$\phi'(0) = 0 \Rightarrow c_1 r_1 + c_2 r_2 = 0$$

$$c_1 r_1 - c_2 r_1 = 0$$

$$r_1 \neq 0 \Rightarrow c_1 = c_2$$

$$\phi'(L) = 0 \Rightarrow c_1 r_1 \exp(r_1 L) + c_2 r_2 \exp(r_2 L) = 0$$

$$c_1 r_1 [\exp(r_1 L) - \exp(r_2 L)] = 0$$

$$r_1 \neq 0, \text{ and } \exp(r_1 L) \neq \exp(r_2 L), \text{ so } c_1 = 0 \Rightarrow c_2 = 0$$

Thus only $\phi(x)=0$ is a solution.

$\lambda=0$ $\phi''(x)=0$

$$\phi(x) = c_1 + c_2 x, \quad \phi'(x) = c_2$$

$$\phi'(0) = 0 \Rightarrow c_2 = 0$$

$$\phi'(L) = 0 \Rightarrow c_2 = 0$$

$\phi(x) = c_1$ is a solution, nontrivial if $c_1 \neq 0$.

Let's call this nontrivial solution $\phi_0(x) = c_1, \lambda=0$

$$\text{If } \lambda=0 \rightarrow T_0'(t) = -\lambda k T_0(t) = 0$$

$$T_0'(t) = c_2$$

So this solution is $u_0(x,t) = \phi_0(x) T_0(t) = c$, for $c \neq 0$.

$\lambda > 0$ $\phi'' + |\lambda| \phi = 0$

$$r^2 + |\lambda| = 0 \Rightarrow r_1 = i\sqrt{|\lambda|}, r_2 = -i\sqrt{|\lambda|}$$

$$\phi(x) = c_1 \cos(x\sqrt{|\lambda|}) + c_2 \sin(x\sqrt{|\lambda|})$$

$$\phi'(x) = c_1 \sqrt{|\lambda|} (-\sin(x\sqrt{|\lambda|})) + c_2 \sqrt{|\lambda|} \cos(x\sqrt{|\lambda|})$$

$$\phi'(0) = 0 \Rightarrow c_2 \sqrt{|\lambda|} = 0 \Rightarrow c_2 = 0$$

$$\phi'(L) = 0 \Rightarrow -c_1 \sqrt{|\lambda|} \sin(L\sqrt{|\lambda|}) = 0 \Rightarrow \sin(L\sqrt{|\lambda|}) = 0$$

$$L\sqrt{|\lambda|} = n\pi, \quad n=1,2,3,\dots$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

$$\lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2 \Rightarrow \phi_n(x) = \cos\left(\frac{n\pi x}{L}\right)$$

$$\lambda = \lambda_n \Rightarrow T_n'(t) = -\lambda_n k T_n(t)$$

$$T_n(t) = c_n \exp(-\lambda_n k t)$$

These solutions are $u_n(x,t) = \phi_n(x) T_n(t) = c_n \exp(-\lambda_n k t) \cos\left(\frac{n\pi x}{L}\right)$
 $\lambda_n = \left(\frac{n\pi}{L}\right)^2$

For $t=0$, $u_0(x,t)$ does not grow in time

For $t > 0$, $u_n(x,t)$ decays exponentially in time

So no separated solutions grow exponentially in time.

(c) General solution is $u(x,t) = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t)$

$$= A_0 + \sum_{n=1}^{\infty} A_n \exp(-k(\frac{n\pi}{L})^2 t) \cos(\frac{n\pi x}{L})$$

This satisfies initial condition of $u(x,0) = f(x)$

$$A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{L}) = f(x)$$

(d) Use $\int_0^L \cos(\frac{n\pi x}{L}) \cos(\frac{m\pi x}{L}) dx = \begin{cases} 0, & n \neq m \\ \frac{L}{2}, & n = m \neq 0 \\ L, & n = m = 0. \end{cases}$

$$A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{L}) = f(x)$$

↓ multiply by $\cos(\frac{m\pi x}{L})$, integrate

$$A_0 L = \int_0^L f(x) dx \quad (\text{if } m=0)$$

$$A_m \frac{L}{2} = \int_0^L f(x) \cos(\frac{m\pi x}{L}) dx \quad (\text{if } m \neq 0)$$

$$A_0 = \frac{1}{L} \int_0^L f(x) dx$$

$$A_m = \frac{2}{L} \int_0^L f(x) \cos(\frac{m\pi x}{L}) dx$$

(e) the equilibrium/steady-state solution of the PDE is $u(x) = \frac{1}{L} \int_0^L f(x) dx$.

Our solution is $u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \exp(-k(\frac{n\pi}{L})^2 t) \cos(\frac{n\pi x}{L})$

Taking $t \rightarrow \infty$: $\lim_{t \rightarrow \infty} u(x,t) = A_0 = \frac{1}{L} \int_0^L f(x) dx$ ✓.

(4)

2.3.P

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} - \alpha u$$

$$u(0,t) = 0, \quad u(L,t) = 0.$$

(a) $\alpha > 0$, $\frac{\partial u}{\partial t} = 0 \implies u''(x) - \frac{\alpha}{k} u(x) = 0$, $\frac{\alpha}{k} > 0$.

$$r^2 - \frac{\alpha}{k} = 0$$

$$r_1 = \sqrt{\frac{\alpha}{k}}, \quad r_2 = -\sqrt{\frac{\alpha}{k}}, \quad r_1 \neq r_2$$

$$u(x) = c_1 \exp(r_1 x) + c_2 \exp(r_2 x)$$

$$u(0) = 0 \implies c_1 + c_2 = 0, \text{ so } c_1 = -c_2$$

$$u(L) = 0 \implies c_1 \exp(r_1 L) - c_1 \exp(r_2 L) = 0$$

$$r_1 \neq r_2 \implies \exp(r_1 L) - \exp(r_2 L) \neq 0 \implies c_1 = 0$$

$$c_2 = 0$$

Equilibrium solution is $u(x) = 0$.

(b) Separation of variables: $u(x,t) = \phi(x)T(t)$

$$u_t = k u_{xx} - \alpha u$$

$$\frac{T'(t)}{kT(t)} = \frac{\phi''(x)}{\phi(x)} - \frac{\alpha}{k} = -\lambda \quad (\text{unknown constant})$$

$$\text{ODE for } \phi: \begin{cases} \phi''(x) + (\lambda - \frac{\alpha}{k}) \phi(x) = 0 \\ \phi(0) = 0, \quad \phi(L) = 0 \end{cases}$$

We already know the nontrivial solutions for this problem:

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad \text{with } (\lambda_n - \frac{\alpha}{k}) = \left(\frac{n\pi}{L}\right)^2$$

$$\lambda_n = \frac{\alpha}{k} + \left(\frac{n\pi}{L}\right)^2$$

For $\lambda = \lambda_n$: $T_n'(t) = -k \lambda_n T_n(t)$
 $= -(\alpha + k(\frac{n\pi}{L})^2) T_n(t)$

$T_n(t) = \exp(-(\alpha + k(\frac{n\pi}{L})^2) t)$

$u(x,t) = \sum_{n=1}^{\infty} A_n \phi_n(x) T_n(t) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{L}) \exp(-(\alpha + k(\frac{n\pi}{L})^2) t)$

Since $\alpha + k(\frac{n\pi}{L})^2 > 0$, then $\lim_{t \rightarrow \infty} u(x,t) = 0$, which matches (a).

12.4.1

$u_t = k u_{xx}$, $0 < x < L$, $t > 0$

$\frac{\partial u}{\partial t}(0,t) = 0$, $\frac{\partial u}{\partial t}(L,t) = 0$

$u(x,0) = f(x)$

Separation of variables: $u(x,t) = \phi(x) T(t)$

$u_t = k u_{xx}$

$\frac{T'(t)}{k T(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda$

$\begin{cases} \phi'' + \lambda \phi = 0 \\ \phi'(0) = 0, \phi'(L) = 0 \end{cases}$

Table in book $\Rightarrow \lambda_n = (\frac{n\pi}{L})^2$, $\phi_n(x) = \cos(\frac{n\pi x}{L})$, $n = 0, 1, 2, \dots$

$T_n'(t) = -\lambda_n k T_n(t)$

$T_n(t) = \exp(-k \lambda_n t) = \exp(-k (\frac{n\pi}{L})^2 t)$

$u(x,t) = \sum_{n=0}^{\infty} A_n \exp(-k (\frac{n\pi}{L})^2 t) \cos(\frac{n\pi x}{L})$

@ $t=0$: $f(x) = \sum_{n=0}^{\infty} A_n \cos(\frac{n\pi x}{L})$

Orthogonality: $\int_0^L \cos(\frac{n\pi x}{L}) \cos(\frac{m\pi x}{L}) dx = \begin{cases} 0, & m \neq n \\ L, & m = n = 0 \\ L/2, & m = n \neq 0 \end{cases}$

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \text{orthogonality} \Rightarrow A_0 = \frac{1}{L} \int_0^L f(x) dx \quad (6)$$

$$A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n > 0$$

$$(a) u(x,0) = \begin{cases} 0, & x < L/2 \\ 1, & x > L/2 \end{cases}$$

$$A_0 = \frac{1}{L} \int_0^L f(x) dx = \frac{1}{L} \int_{L/2}^L 1 dx = \frac{1}{2}$$

$$A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L} \int_{L/2}^L \cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L} \cdot \frac{L}{n\pi} \sin\left(\frac{n\pi x}{L}\right) \Big|_{L/2}^L$$

$$= \frac{2}{n\pi} \left[\sin(n\pi) - \sin\left(n\frac{\pi}{2}\right) \right]$$

$$= \frac{2}{n\pi} \left(0 - \sin\left(\frac{n\pi}{2}\right) \right)$$

$$= \begin{cases} 0, & n \text{ even} \\ (-1)^{\frac{n+1}{2}} \frac{2}{n\pi}, & n \text{ odd} \end{cases}$$

$$u(x,t) = \sum_{n=0}^{\infty} A_n \exp(-k_n t) \cos\left(\frac{n\pi x}{L}\right)$$

$$= \frac{1}{2} + \sum_{n \text{ odd}} (-1)^{\frac{n+1}{2}} \frac{2}{n\pi} \exp(-k_n t) \cos\left(\frac{n\pi x}{L}\right)$$

$$u(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n 2}{\pi(2n-1)} \exp\left(-k \left(\frac{(2n-1)\pi}{L}\right)^2 t\right) \cos\left(\frac{(2n-1)\pi x}{L}\right)$$

$$(b) u(x,0) = f(x) = 6 + 4 \cos\left(\frac{3\pi x}{L}\right)$$

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$$A_0 = \frac{1}{L} \int_0^L f(x) dx = \frac{1}{L} \int_0^L \left(6 + 4 \cos\left(\frac{3\pi x}{L}\right)\right) dx \\ = 6$$

$$A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L} \int_0^L 6 \cos\left(\frac{n\pi x}{L}\right) dx + \frac{2}{L} \int_0^L 4 \cos\left(\frac{3\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx \\ = \begin{cases} 0, & n \neq 3 \text{ (orthogonality)} \\ 4, & n = 3 \end{cases}$$

$$u(x,t) = \sum_{n=0}^{\infty} A_n \exp(-\lambda_n k t) \cos\left(\frac{n\pi x}{L}\right)$$

$$\underline{u(x,t) = 6 + 4 \exp\left(-k \left(\frac{3\pi}{L}\right)^2 t\right) \cos\left(\frac{3\pi x}{L}\right)}$$

2.4.3 $\varphi'' + \lambda \varphi = 0$

$$\varphi(0) = \varphi(2\pi), \quad \varphi'(0) = \varphi'(2\pi)$$

$$\underline{\lambda < 0} \quad \varphi(x) = c_1 \exp(r_1 x) + c_2 \exp(r_2 x), \quad r_1 = \sqrt{|\lambda|}, \quad r_2 = -\sqrt{|\lambda|} = -r_1$$

$$\varphi(0) = \varphi(2\pi) \Rightarrow c_1 + c_2 = c_1 \exp(r_1 2\pi) + c_2 \exp(-r_1 2\pi)$$

$$c_1 [1 - \exp(2\pi r_1)] + c_2 [1 - \exp(-2\pi r_1)] = 0$$

$$\varphi'(0) = \varphi'(2\pi) \Rightarrow c_1 r_1 - c_2 r_1 = c_1 r_1 \exp(2\pi r_1) - c_2 r_1 \exp(-2\pi r_1)$$

$$c_1 r_1 [1 - \exp(2\pi r_1)] + c_2 r_1 [-1 + \exp(-2\pi r_1)] = 0$$

$$\text{Let } A = 1 - \exp(2\pi r_1)$$

$$B = 1 - \exp(-2\pi r_1) \quad (\text{note } A \neq B)$$

$$A \neq 0 \text{ and } B \neq 0 \text{ since } r_1 \neq 0$$

Then our equations are

$$Ac_1 + Bc_2 = 0$$

$$Ac_1 - Bc_2 = 0$$

$$\rightarrow \begin{pmatrix} A & B \\ A & -B \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

determinant of this matrix is $-2AB \neq 0$

So the only solution is $c_1 = c_2 = 0$. \rightarrow no negative λ are eigenvalues.

$\lambda = 0$: $\phi(x) = c_1 + c_2 x$

$$\phi(0) = \phi(2\pi) \Rightarrow c_1 = c_1 + 2\pi c_2 \Rightarrow c_2 = 0.$$

$$\phi'(0) = \phi'(2\pi) \Rightarrow c_2 = c_2 = 0$$

So $\lambda_0 = 0$ with $\phi_0(x) = 1$ is an eigenvalue/eigenfunction.

$\lambda > 0$: $\phi(x) = c_1 \cos(x\sqrt{|\lambda|}) + c_2 \sin(x\sqrt{|\lambda|})$

$$\phi(0) = \phi(2\pi) \Rightarrow c_1 = c_1 \cos(2\pi\sqrt{|\lambda|}) + c_2 \sin(2\pi\sqrt{|\lambda|})$$

$$\phi'(0) = \phi'(2\pi) \Rightarrow \sqrt{|\lambda|} c_2 = -c_1 \sqrt{|\lambda|} \sin(2\pi\sqrt{|\lambda|}) + c_2 \sqrt{|\lambda|} \cos(2\pi\sqrt{|\lambda|})$$

$$\text{Let } A = \cos(2\pi\sqrt{|\lambda|}), \quad B = \sin(2\pi\sqrt{|\lambda|})$$

Our equations are $c_1 = Ac_1 + Bc_2$
 $c_2 = -Bc_1 + Ac_2 \rightarrow \begin{pmatrix} A-1 & B \\ -B & A-1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{aligned} \text{Determinant of matrix is } (A-1)^2 + B^2 &= A^2 + B^2 + 1 - 2A \\ &= 2 - 2A \end{aligned}$$

This equals 0 when $A = 1$

$$\cos(2\pi\sqrt{|\lambda|}) = 1$$

$$\lambda_n = n^2, \quad n = 1, 2, \dots$$

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If $\lambda = \lambda_n$: $A = \cos(2\pi n) = 1$
 $B = \sin(2\pi n) = 0$

Then $\begin{pmatrix} A-1 & B \\ -B & A-1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

So c_1 and c_2 can be anything.

Thus $\lambda = \lambda_n \Rightarrow C_n = \cos(nx)$ is an eigenfunction
 $S_n = \sin(nx)$ is another eigenfunction

λ	ϕ
0	1
$n^2 (n > 0)$	$\cos(nx)$
$n^2 (n > 0)$	$\sin(nx)$

12.4.4 $\phi'' + \lambda \phi = 0$

$\phi'(0) = 0, \phi'(L) = 0$

$\lambda < 0$: $\phi(x) = c_1 \exp(r_1 x) + c_2 \exp(r_2 x), \quad r_1 = \sqrt{|\lambda|}, \quad r_2 = -r_1 = -\sqrt{|\lambda|}$

$\phi'(x) = c_1 r_1 \exp(r_1 x) - c_2 r_1 \exp(-r_1 x)$

$\phi'(0) = 0 \Rightarrow c_1 - c_2 = 0 \Rightarrow c_1 = c_2$

$\phi'(L) = 0 \Rightarrow c_1 \exp(r_1 L) - c_2 \exp(-r_1 L) = 0$

$c_1 [\exp(r_1 L) - \exp(-r_1 L)] = 0$

$\neq 0$ since $r_1 \neq 0$.

So $c_1 = 0 \Rightarrow c_2 = 0 \Rightarrow \lambda < 0$ not an eigenvalue.