

2.3.1 (a) $\frac{\partial u}{\partial t} = \frac{k}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right)$

suppose $u = u(t, r) = T(t) R(r)$

$$\text{PDE} \rightarrow \frac{\partial}{\partial t} (T(t) R(r)) = \frac{k}{r} \frac{\partial}{\partial r} \left[r \frac{\partial}{\partial r} (T(t) R(r)) \right]$$

$$\begin{aligned} R \cdot \frac{dT}{dt} &= T \cdot \frac{k}{r} \frac{\partial}{\partial r} \left(r \frac{\partial R}{\partial r} \right) \\ &= T \cdot \frac{k}{r} \left(\frac{\partial R}{\partial r} + r \frac{\partial^2 R}{\partial r^2} \right) \end{aligned}$$

$$R \frac{dT}{dt} = k \cdot \frac{T}{r} \frac{dR}{dr} + T \cdot k \frac{d^2 R}{dr^2}$$

$$\frac{T'(t)}{T} = k \cdot \frac{R'(r)}{r \cdot R} + k \cdot \frac{R''(r)}{R} = -\lambda \rightarrow \text{constant because LHS is only a function of } t, \text{ RHS is only a function of } r. \text{ (negative sign is arbitrary)}$$

$$\begin{cases} T'(t) + \lambda T(t) = 0 \\ R''(r) + \frac{1}{r} R'(r) + \frac{\lambda}{k} R(r) = 0 \end{cases}$$

(c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$u(x, y) = \phi(x) \psi(y)$$

$$\text{PDE} \rightarrow \frac{\partial^2}{\partial x^2} (\phi(x) \psi(y)) + \frac{\partial^2}{\partial y^2} (\phi(x) \psi(y)) = 0$$

$$\phi'' \cdot \psi + \phi \psi'' = 0$$

$$\frac{\phi''}{\phi} = -\frac{\psi''}{\psi} = \lambda \quad (\text{constant})$$

$$\begin{cases} \phi''(x) - \lambda \phi(x) = 0 \\ \psi''(y) + \lambda \psi(y) = 0 \end{cases}$$

$$(e) \quad \frac{\partial u}{\partial t} = k \frac{\partial^4 u}{\partial x^2}$$

(2)

$$u(x, t) = \phi(x) T(t)$$

$$\text{PDE} \rightarrow \frac{\partial}{\partial t}(\phi T) = k \frac{\partial^4}{\partial x^4}(\phi T)$$

$$\phi T' = k \phi^{(4)} T$$

$$\frac{T'}{T} = k \frac{\phi^{(4)}}{\phi} = \lambda \text{ (constant)}$$

$$\begin{cases} T'(t) - \lambda T(t) = 0 \\ \phi^{(4)}(x) - \frac{\lambda}{k} \phi(x) = 0 \end{cases}$$

12.3.2 (a) $\phi''(x) + \lambda \phi(x) = 0$

$$\phi(0) = 0, \quad \phi(\pi) = 0$$

$\lambda < 0$: assume $\phi(x) = \exp(rx)$ for r an unknown constant.
 $\phi''(x) = r^2 \exp(rx)$

$$\text{ODE} \rightarrow r^2 \exp(rx) + \lambda \exp(rx) = 0$$

"characteristic equation" $\rightarrow r^2 = -\lambda$ ($\exp(rx)$ never vanishes)

$$r^2 = |\lambda| \text{ (since } \lambda < 0)$$

$$r = \pm \sqrt{|\lambda|}. \text{ Let } r_1 = +\sqrt{|\lambda|}, \quad r_2 = -\sqrt{|\lambda|}$$

$$\phi(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

$$\phi(0) = 0 \Rightarrow c_1 + c_2 = 0 \longrightarrow c_1 = -c_2$$

$$\phi(\pi) = 0 \Rightarrow c_1 e^{\pi r_1} + c_2 e^{\pi r_2} = 0.$$

$$c_2 [e^{\pi r_2} - e^{\pi r_1}] = 0$$

never vanishes since $r_1 = \sqrt{|\lambda|} \neq -\sqrt{|\lambda|} = r_2$

$$c_2 = 0 \Rightarrow c_1 = 0$$

So there are no nontrivial solutions $\Rightarrow \lambda < 0$ not an eigenvalue

(3)

$\lambda = 0$. Characteristic equation: $r^2 = 0$
 $r = 0$

$$\phi(x) = c_1 + c_2 x$$

$$\phi(0) = 0 \Rightarrow c_1 = 0$$

$$\phi(\pi) = 0 \Rightarrow c_2 \pi = 0 \rightarrow c_2 = 0$$

Again no nontrivial solutions $\rightarrow \lambda = 0$ not an eigenvalue.

$\lambda > 0$ Characteristic equation: $r^2 = -\lambda = -|\lambda|$
 $r = \pm i\sqrt{|\lambda|}$

$$\phi(x) = c_1 \cos(x\sqrt{|\lambda|}) + c_2 \sin(x\sqrt{|\lambda|})$$

$$\phi(0) = 0 \Rightarrow c_1 = 0$$

$$\phi(\pi) = 0 \Rightarrow c_2 \sin(\pi\sqrt{|\lambda|}) = 0$$

c_2 can be nonzero if $\sin(\pi\sqrt{|\lambda|}) = 0$

$$\sqrt{|\lambda|} = 1, 2, 3, \dots$$

Let $\lambda_n = n^2$ for $n = 1, 2, \dots$

$\lambda = \lambda_n \Rightarrow \phi(x) = \sin(x\sqrt{\lambda_n}) = \sin(nx)$ is a solution.

eigenvalues $\lambda_n = n^2$ ($n = 1, 2, \dots$), eigenfunctions $\phi_n(x) = \sin(nx)$

(b) $\phi(0) = 0$, $\phi(1) = 0$

$$\phi''(x) + \lambda \phi(x) = 0$$

$\lambda < 0$: characteristic equation $r^2 = -\lambda = |\lambda|$
 $r = \pm \sqrt{|\lambda|}$

$$\phi(x) = c_1 \exp(x\sqrt{|\lambda|}) + c_2 \exp(-x\sqrt{|\lambda|})$$

$$\phi(0) = 0 \Rightarrow c_1 + c_2 = 0 \longrightarrow c_1 = -c_2$$

$$\phi(1) = 0 \Rightarrow c_1 e^{\sqrt{|\lambda|}} + c_2 e^{-\sqrt{|\lambda|}} = 0$$

$$c_2 (e^{-\sqrt{|\lambda|}} - e^{\sqrt{|\lambda|}}) = 0 \rightarrow c_2 = 0 \rightarrow c_1 = 0$$

So no negative eigenvalues

(9)

$$\underline{\lambda=0} \quad \varphi''(x) = 0$$

$$\varphi(x) = c_1 + c_2 x$$

$$\varphi(0) = 0 \Rightarrow c_1 = 0$$

$$\varphi(1) = 0 \Rightarrow c_2 = 0$$

$\lambda=0$ not an eigenvalue

$$\underline{\lambda > 0} \quad \text{characteristic equation: } r^2 = -\lambda = -|\lambda|$$

$$r = \pm i\sqrt{|\lambda|}$$

$$\varphi(x) = c_1 \cos(x\sqrt{|\lambda|}) + c_2 \sin(x\sqrt{|\lambda|})$$

$$\varphi(0) = 0 \Rightarrow c_1 = 0$$

$$\varphi(1) = 0 \Rightarrow c_2 \sin(\sqrt{|\lambda|}) = 0$$

$$c_2 \neq 0 \text{ if } \sin(\sqrt{|\lambda|}) = 0 \rightarrow \lambda_n = (n\pi)^2, \quad n=1, 2, \dots$$

$$\text{eigenvalues } \lambda_n = (n\pi)^2, \quad n=1, 2, \dots$$

$$\text{eigenfunctions } \varphi_n(x) = \sin(n\pi x)$$

$$(d) \quad \varphi''(x) + \lambda\varphi(x) = 0$$

$$\varphi(0) = 0, \quad \varphi'(L) = 0$$

$$\underline{\lambda < 0}: \text{characteristic equation } r^2 = -\lambda = |\lambda|$$

$$r = \pm\sqrt{|\lambda|}, \quad r_1 = +\sqrt{|\lambda|}, \quad r_2 = -\sqrt{|\lambda|}$$

$$\varphi(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

$$\varphi(0) = 0 \Rightarrow c_1 + c_2 = 0 \longrightarrow c_1 = -c_2$$

$$\varphi'(L) = 0 \Rightarrow c_1 r_1 e^{r_1 L} + c_2 r_2 e^{r_2 L} = 0$$

$$c_2 (r_2 e^{r_2 L} - r_1 e^{r_1 L}) = 0$$

$$c_2 (r_2 e^{r_2 L} + r_2 e^{r_1 L}) = 0 \quad (\text{since } r_1 = -r_2)$$

$$c_2 r_2 (e^{r_2 L} + e^{r_1 L}) = 0$$

\uparrow $r_2 \neq 0$ \uparrow sum of positive numbers $\neq 0$

So $c_2 = 0 \rightarrow c_1 = 0$
 $\lambda < 0$ not an eigenvalue

$\lambda = 0$ $\phi''(x) = 0$

$$\phi(x) = c_1 + c_2 x$$

$$\phi(0) = 0 \Rightarrow c_1 = 0$$

$$\phi'(L) = 0 \Rightarrow c_2 = 0$$

$\lambda = 0$ not an eigenvalue

$\lambda \neq 0$ characteristic equation: $r^2 = -\lambda = -|\lambda|$
 $r = \pm i\sqrt{|\lambda|}$

$$\phi(x) = c_1 \cos(x\sqrt{|\lambda|}) + c_2 \sin(x\sqrt{|\lambda|})$$

$$\phi(0) = 0 \Rightarrow c_1 = 0$$

$$\phi'(L) = 0 \Rightarrow c_2 \sqrt{|\lambda|} \cos(L\sqrt{|\lambda|}) = 0$$

c_2 can be nonzero if $\cos(L\sqrt{|\lambda|}) = 0$

$$L\sqrt{|\lambda|} = (2n-1)\frac{\pi}{2}, n=1, 2, \dots$$

$$\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2, n=1, 2, \dots$$

eigenvalues $\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2, n=1, 2, \dots$

eigenfunctions $\phi_n(x) = \sin\left(x \cdot \frac{(2n-1)\pi}{2L}\right)$.

2.3.3

$$u_t = k u_{xx} \quad (k > 0)$$

$$u(0, t) = 0, \quad u(L, t) = 0$$

Before using initial data, find eigenfunctions via separation of variables.

assume $u(x,t) = \phi(x) T(t)$

(6)

$$u_t = k u_{xx} \rightarrow \frac{T'(t)}{T(t)} = \frac{k \phi''(x)}{\phi(x)} = -\lambda, \text{ (constant)}$$

$$\phi''(x) + \frac{\lambda}{k} \phi(x) = 0$$

$$u(0,t) = 0 \Rightarrow \phi(0) = 0$$

$$u(L,t) = 0 \Rightarrow \phi(L) = 0$$

$$\lambda/k < 0 \Rightarrow \phi(x) = c_1 \exp(x \sqrt{\lambda/k}) + c_2 \exp(-x \sqrt{\lambda/k})$$

$$\phi(0) = 0 \Rightarrow c_1 + c_2 = 0 \rightarrow c_1 = -c_2$$

$$\phi(L) = 0 \Rightarrow c_1 \exp(L \sqrt{\lambda/k}) + c_2 \exp(-L \sqrt{\lambda/k}) = 0$$

$$c_2 [\exp(-L \sqrt{\lambda/k}) - \exp(L \sqrt{\lambda/k})] = 0$$

$\neq 0$ since exponents differ

$c_2 = 0 \rightarrow c_1 = 0 \rightarrow \lambda < 0$ not an eigenvalue

$$\lambda/k = 0 \Rightarrow \phi(x) = c_1 + c_2 x$$

$$\phi(0) = 0 \Rightarrow c_1 = 0$$

$$\phi(L) = 0 \Rightarrow c_2 = 0$$

$\lambda/k = 0$ not an eigenvalue.

$$\lambda/k > 0 \Rightarrow \phi(x) = c_1 \cos(x \sqrt{\lambda/k}) + c_2 \sin(x \sqrt{\lambda/k})$$

$$\phi(0) = 0 \Rightarrow c_1 = 0$$

$$\phi(L) = 0 \Rightarrow c_2 \sin(L \sqrt{\lambda/k}) = 0$$

$$L \sqrt{\lambda/k} = n\pi, \quad n=1, 2, 3, \dots$$

$$\lambda_n = k \left(\frac{n\pi}{L} \right)^2, \quad n=1, 2, \dots$$

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n=1, 2, \dots$$

When $k = \lambda_n$: $T'(t) + \lambda_n T(t) = 0$

$$T_n(t) = c_3 \exp(-\lambda_n t) = c_3 \exp\left(-k \left(\frac{n\pi}{L}\right)^2 t\right), \quad n=1, 2, \dots$$

$$u = u_n(x, t) = \phi_n(x) T_n(t) = b_n \exp(-k \left(\frac{n\pi}{L}\right)^2 t) \sin\left(\frac{n\pi x}{L}\right)$$

solves PDE for any constant b_n

Superposition: $u(x, t) = \sum_{n=1}^{\infty} b_n \exp(-k \left(\frac{n\pi}{L}\right)^2 t) \sin\left(\frac{n\pi x}{L}\right)$ solves PDE.

$$\text{At } t=0: u(x, 0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

(a) $u(x, 0) = 6 \sin\left(\frac{9\pi x}{L}\right)$

Find constants b_n such that $6 \sin\left(\frac{9\pi x}{L}\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$

Can do this by inspection, but instead use orthogonality:

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0, & m \neq n \\ L/2, & m = n \end{cases} \quad (\text{see, eg. problem 5}).$$

$$6 \sin\left(\frac{9\pi x}{L}\right) \stackrel{?}{=} \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\int_0^L 6 \sin\left(\frac{9\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \int_0^L \left(\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \right) \sin\left(\frac{m\pi x}{L}\right) dx$$

$$6 \int_0^L \sin\left(\frac{9\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \sum_{n=1}^{\infty} b_n \underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx}_{\text{only nonzero if } m=n}$$
$$= b_m \cdot L/2$$

$$b_m \cdot L/2 = 6 \int_0^L \sin\left(\frac{9\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0, & m \neq 9 \\ 6L/2, & m = 9 \end{cases}$$

$$\text{So } b_m = \begin{cases} 0, & m \neq 9 \\ 6, & m = 9 \end{cases}$$

Full solution: $u(x, t) = \sum_{n=1}^{\infty} b_n \exp(-k \left(\frac{n\pi}{L}\right)^2 t) \sin\left(\frac{n\pi x}{L}\right) = 6 \exp(-k \left(\frac{9\pi}{L}\right)^2 t) \sin\left(\frac{9\pi x}{L}\right)$

(8)

$$(c) u(x,0) = 2 \cos\left(\frac{3\pi x}{L}\right)$$

$$\text{Find } b_n \text{ such that } 2 \cos\left(\frac{3\pi x}{L}\right) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$2 \int_0^L \sin\left(\frac{m\pi x}{L}\right) \cos\left(\frac{3\pi x}{L}\right) dx = \sum_{n=1}^{\infty} b_n \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx$$
$$= b_m \cdot \frac{L}{2}$$

$$b_m = \frac{4}{L} \int_0^L \sin\left(\frac{m\pi x}{L}\right) \cos\left(\frac{3\pi x}{L}\right) dx$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \exp\left(-k \left(\frac{n\pi}{L}\right)^2 t\right) \sin\left(\frac{n\pi x}{L}\right), \text{ where } b_n = \frac{4}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{3\pi x}{L}\right) dx$$

2.3.5 $\sin a \sin b = \frac{1}{2} [\cos(a-b) - \cos(a+b)]$

$$\text{Then } \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \frac{1}{2} \int_0^L \cos\left(\frac{(n-m)\pi x}{L}\right) dx - \frac{1}{2} \int_0^L \cos\left(\frac{(n+m)\pi x}{L}\right) dx$$

$$\text{If } m \neq n: \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \frac{1}{2} \cdot \frac{L}{\pi(n-m)} \sin\left(\frac{(n-m)\pi x}{L}\right) \Big|_0^L - \frac{1}{2} \cdot \frac{L}{\pi(n+m)} \sin\left(\frac{(n+m)\pi x}{L}\right) \Big|_0^L$$
$$= \frac{L}{2\pi(n-m)} (0 - 0) - \frac{L}{2\pi(n+m)} (0 - 0) = 0$$

$$\text{If } m = n: \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \frac{1}{2} \int_0^L 1 dx - \frac{1}{2} \int_0^L \cos\left(\frac{(n+m)\pi x}{L}\right) dx$$
$$= \frac{L}{2} - \frac{L}{2\pi(n+m)} \sin\left(\frac{(n+m)\pi x}{L}\right) \Big|_0^L$$
$$= \frac{L}{2}$$

$$\text{So } \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0, & m \neq n \\ \frac{L}{2}, & m = n \end{cases}$$