1.4.1 (a) Heat equation: $c\rho \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + Q$

Equilibrium $\Rightarrow \frac{\partial u}{\partial t} = 0$, heat equation becomes $\frac{d^2u}{dx^2} = -Q/K_0$.

$Q=0 \Rightarrow \frac{d^2u}{dx^2} = 0, \quad u(0)=0, \quad u(L)=T$

$\frac{du}{dx} = c_1$

$u(x) = c_1 x + c_2$

$u(0)=0 \Rightarrow 0 = c_1 \cdot 0 + c_2$

$c_2 = 0$

$u(L)=T \Rightarrow T = c_1 \cdot L + c_2$

$T = c_1 L \rightarrow c_1 = T/L$

$u(x) = \frac{Tx}{L}$

(b) $Q=0$, $u(0)=T$, $u(L)=0 \Rightarrow \frac{d^2u}{dx^2} = 0, \quad u(0)=T \quad u(L)=0$

$u(x) = c_1 x + c_2$

$u(0)=T \Rightarrow c_2 = T$

$u(L)=0 \Rightarrow 0 = c_1 L + T \rightarrow c_1 = -T/L$

$u(x) = T - \frac{Tx}{L}$

(c) $Q=0$, $\frac{\partial u}{\partial x}(0)=0$, $u(L)=T \Rightarrow \frac{d^2u}{dx^2} = 0, \quad u'(0)=0, \quad u(L)=T$

$u(x) = c_1 x + c_2$

$u'(0)=0 \Rightarrow c_1 = 0$

$u(L)=T \Rightarrow c_2 = T$

$u(x) = T$

(d) $Q=0$, $u(0)=T$, $\frac{\partial u}{\partial x}(L)=\alpha \Longrightarrow \frac{d^2u}{dx^2}=0$, $u(0)=T$, $u'(L)=\alpha$

$$u(x)=c_1x+c_2$$
$$u(0)=T \Rightarrow c_2=T$$
$$u'(L)=\alpha \Rightarrow \alpha=c_1$$

$$u(x)=\alpha x+T$$

(e) $\frac{Q}{K_0}=1$, $u(0)=T_1$, $u(L)=T_2 \Longrightarrow \frac{d^2u}{dx^2}=-1$, $u(0)=T_1$, $u(L)=T_2$

$$u(x)=-\frac{x^2}{2}+c_1x+c_2$$
$$u(0)=T_1 \Rightarrow T_1=-0+c_1\cdot 0+c_2 \longrightarrow c_2=T_1$$
$$u(L)=T_2 \Rightarrow T_2=-\frac{L^2}{2}+c_1L+c_2$$
$$c_1=\frac{1}{L}\left[T_2-T_1+\frac{L^2}{2}\right]$$

$$u(x)=-\frac{x^2}{2}+\frac{1}{L}\left[T_2-T_1+\frac{L^2}{2}\right]+T_1$$

1.4.5 Equilibrium with no sources: $\frac{d^2u}{dx^2}=0$ $u(L)=T$, T is unknown.

Known temperature @ $x=0 \to u(0)=T_0$ (known)

known heat flow @ $x=0 \to -K_0\frac{du}{dx}(0)=\phi_0$ (known)

$$u''=0,\quad u(0)=T_0,\quad -K_0u'(0)=\phi_0$$
$$u(x)=c_1x+c_2$$
$$u(0)=T_0 \Longrightarrow c_2=T_0$$
$$-K_0u'(0)=\phi_0 \Longrightarrow -K_0c_1=\phi_0 \to c_1=-\phi_0/K_0$$

$$u(x)=T_0-\frac{\phi_0}{K_0}x$$

$$u(L)=T \Rightarrow \underline{T=T_0-\frac{\phi_0 L}{K_0}}$$

1.4.7 (a) Equilibrium $\Rightarrow \frac{\partial u}{\partial t}=0$ ③

$$0=\frac{d^2u}{dx^2}+1, \quad u(x,0)=f(x)$$

$$u'(0)=1, \quad u'(L)=\beta.$$

$$u(x)=\frac{-x^2}{2}+c_1x+c_2$$

$$u'(0)=1 \Rightarrow 1=0+c_1 \to c_1=1$$

$$u'(L)=\beta \Rightarrow \beta=-L+c_1=-L+1, \quad \underline{\beta=1-L}$$

$$u(x)=-\frac{x^2}{2}+x+c_2$$

This value of β (flux of heat at $x=L$) balances energy source terms and fluxes at boundaries.

Conservation of energy: at steady-state, energy in system should be conserved.

energy @ $t=0$: $\int_0^L u(x,0)dx=\int_0^L f(x)dx$ ($c\rho=1$ from equation)

at steady-state: $\int_0^L u(x)\,dx=\int_0^L\left(-\frac{x^2}{2}+x+c_2\right)dx$

$$=-\frac{L^3}{6}+\frac{L^2}{2}+c_2L=\int_0^L f(x)dx$$

$$c_2=\frac{1}{L}\left[\int_0^L f(x)dx+\frac{L^3}{6}-\frac{L^2}{2}\right]$$

$$u(x)=-\frac{x^2}{2}+x+\frac{1}{L}\left(\int_0^L f(x)dx+\frac{L^3}{6}-\frac{L^2}{2}\right)$$

(b) Equilibrium $\to \frac{\partial u}{\partial t}=0$

$$\frac{d^2u}{dx^2}=0 \to u(x)=c_1+c_2x$$

$$\left.\begin{array}{l} u'(0)=1 \Rightarrow c_2=1 \\ u'(L)=\beta \Rightarrow c_2=\beta \end{array}\right\}\beta=1$$

There are no sources in this PDE. $\beta=1$ ensures that heat flux out equals that in $\to$ i.e. conservation of energy.

To determine constant C_1: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

$$\int_0^L \frac{\partial u}{\partial t}\,dx = \int_0^L \frac{\partial^2 u}{\partial x^2}\,dx$$

$$\frac{\partial}{\partial t}\left(\int_0^L u\,dx\right) = \frac{\partial u}{\partial x}\Big|_0^L = \frac{\partial u}{\partial x}(L,t) - \frac{\partial u}{\partial x}(0,t) = \beta - 1 = 0$$

so $\int_0^L u\,dx$ is constant for all time

at $t=0$: $\int_0^L u(x,0)\,dx = \int_0^L f(x)\,dx$

at equilibrium: $\int_0^L u(x)\,dx = \int_0^L (C_1 + x)\,dx = C_1 L + \frac{L^2}{2}$

$$C_1 L + \frac{L^2}{2} = \int_0^L f(x)\,dx$$

$$C_1 = \frac{1}{L}\left(\int_0^L f(x)\,dx - \frac{L^2}{2}\right) \longrightarrow u(x) = x + \frac{1}{L}\left(\int_0^L f(x)\,dx - \frac{L^2}{2}\right)$$

(c) $\frac{\partial u}{\partial t} = 0 \rightarrow \frac{d^2 u}{dx^2} + x - \beta = 0$

$$u'' = \beta - x$$

$$u' = \beta x - \frac{x^2}{2} + c_1$$

$$u = \frac{\beta}{2}x^2 - \frac{x^3}{6} + c_1 x + c_2$$

$$u'(x) = \beta x - \frac{x^2}{2} + c_1 \longrightarrow u'(0) = 0$$

$$\beta \cdot 0 - \frac{0^2}{2} + c_1 = 0$$

$$c_1 = 0$$

$$u'(L) = 0$$

$$\beta L - \frac{L^2}{2} = 0$$

$\beta = \frac{L}{2}$ This value of β ensures that the net energy imparted to the system from the source term is 0.

$$u(x) = \frac{\beta}{2}x^2 - \frac{x^3}{6} + c_2$$

To determine c_2:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x - \beta$$

$$\frac{\partial}{\partial t}\int_0^L u\,dx = \int_0^L \frac{\partial^2 u}{\partial x^2}\,dx + \int_0^L (x-\beta)\,dx$$

$$= \frac{\partial u}{\partial x}(L,t) - \frac{\partial u}{\partial x}(0,t) + \frac{x^2}{2}\Big|_0^L - \beta L$$

$$= 0 - 0 + \frac{L^2}{2} - \frac{L}{2}\cdot L = 0$$

Again, $\int_0^L u(x,t)\,dx$ is constant in t.

At $t=0$: $\int_0^L u(x,0)\,dx = \int_0^L f(x)\,dx$

At equilibrium: $\int_0^L \left(\frac{\beta}{2}x^2 - \frac{x^3}{6} + c_2\right)dx = \frac{1}{6}\cdot\frac{L}{2}\cdot L^3 - \frac{L^4}{24} + c_2 L = \frac{L^4}{24} + c_2 L$

$$c_2 = \frac{1}{L}\left(\int_0^L f(x)\,dx - \frac{L^4}{24}\right)$$

$$u(x) = \frac{L}{4}x^2 - \frac{x^3}{6} + \frac{1}{L}\left(\int_0^L f(x)\,dx - \frac{L^4}{24}\right)$$

2.2.3 $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + \alpha(x,t)u + \beta(x,t)$

Let $f = \beta(x,t)$

Let $L(u) = \frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} - \alpha(x,t)u$

Then the PDE is of the form $Lu = f$.

Is L linear? Let u_1 and u_2 be functions, c_1 and c_2 be constants:

$$L(c_1u_1 + c_2u_2) = \frac{\partial}{\partial t}(c_1u_1 + c_2u_2) - k\frac{\partial^2}{\partial x^2}(c_1u_1 + c_2u_2) - \alpha(x,t)[c_1u_1 + c_2u_2]$$

$$= c_1\frac{\partial u_1}{\partial t} - k\frac{\partial^2 u_1}{\partial x^2}c_1 - \alpha(x,t)c_1u_1 + c_2\frac{\partial u_2}{\partial t} - c_2k\frac{\partial^2 u_2}{\partial x^2} - c_2\alpha(x,t)u_2$$

2.2.4 (a) $L(u_p)=f$, $L(u_1)=0$, $L(u_2)=0$, L is linear.

Then $L(u_p+c_1u_1+c_2u_2) = L(u_p)+c_1L(u_1)+c_2L(u_2)$

$= f+0+0 = f$

Hence $u_p+c_1u_1+c_2u_2$ is a particular solution.

(b) $L(u)=f_1+f_2$, and $L(u_1)=f_1$, $L(u_2)=f_2$, L is linear.

Then $L(u_1+u_2)=L(u_1)+L(u_2)=f_1+f_2$

So u_1+u_2 is a particular solution for $L(u)=f_1+f_2$.

2.2.5 L is linear $\Longrightarrow$ $L(c_1u_1+c_2u_2) = c_1L(u_1)+c_2L(u_2)$

for c_1, c_2 constants and u_1, u_2 functions.

If $M\geq 2$: $L\left(\sum_{m=1}^{M} c_mu_m\right) = c_1L(u_1)+L\left(\sum_{m=2}^{M} c_mu_m\right)$

$= c_1L(u_1)+c_2L(u_2)+L\left(\sum_{m=3}^{M} c_mu_m\right)$

$= c_1L(u_1)+c_2L(u_2)+c_3L(u_3)+L\left(\sum_{m=4}^{M} c_mu_m\right)$

$= \cdots$

$= c_1L(u_1)+c_2L(u_2)+\cdots+c_{M-1}u_{M-1}+L(c_Mu_M)$

$= c_1L(u_1)+c_2L(u_2)+\cdots+c_ML(u_M)$

$= \sum_{m=1}^{M} c_mL(u_m).$

Now suppose that $u_1, u_2, \dots u_M$ are solutions to the linear, homogeneous problem $L(u)=0$. I.e., $L(u_m)=0$ for each $m=1,2,\dots,M$.

Then $L\left(\sum_{m=1}^{M} c_mu_m\right) = \sum_{m=1}^{M} c_mL(u_m) = \sum_{m=1}^{M} c_m\cdot 0 = 0$, so $\sum_{m=1}^{M} c_mu_m$ is a solution.

Thus, superposition extends to arbitrary finite linear combinations.