MATH 3150 HW #10 Solutions

Section 16.4 # 3, 4, 6, 7(a,b,c)

<u>10.4.3</u> (a) $u_t = k u_{xx} + c u_x$, $-\infty < x < \infty$

$u(x,0) = f(x)$

$U(w,t) = \mathcal{F}[u(x,t)]$

$U_t = k\cdot(-iw)^2 U + c(-iw)U$

$= (-iwc - kw^2)U$

$U(w,t) = A(w)\exp(-iwct - kw^2 t)$, $A(w)$: unknown "constant"

Initial condition: $u(x,0) = f(x)$

$U(w,0) = \mathcal{F}[f(x)] = F(w)$

$U(w,0) = A(w)\cdot\exp(0) = F(w) \Rightarrow A(w) = F(w)$

$$U(w,t) = F(w)\underbrace{\exp(-kw^2t)}_{H(w,t)}\exp(-iwct)$$

Convolution theorem: $\mathcal{F}^{-1}[F(w)H(w)] = f(x) * h(x)$,

where $h(x) = \mathcal{F}^{-1}[H(w)]$

$H(w)$ is a Gaussian $\Rightarrow h(x) = \sqrt{\frac{\pi}{kt}}\exp(-x^2/4kt)$

So $\mathcal{F}^{-1}[F(w)H(w)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\sqrt{\frac{\pi}{kt}}\exp(-(x-s)^2/4kt)\,ds$

$u(x,t) = \mathcal{F}^{-1}[U(w,t)] = \mathcal{F}^{-1}[F(w)H(w,t)\cdot\exp(-iwct)]$

Shift property $\Rightarrow \mathcal{F}^{-1}[U(w,t)]$ equals $\mathcal{F}^{-1}[F(w)H(w,t)]$ evaluated at $x+ct$.

So solution is $\frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\sqrt{\frac{\pi}{kt}}\exp\left(-(x-s)^2/4kt\right)ds$ evaluated

at $x \to x+ct$

$$u(x,t)=\int_{-\infty}^{\infty} f(s)\frac{1}{\sqrt{4\pi kt}}\exp\left(-(x+ct-s)^2/4kt\right)ds$$

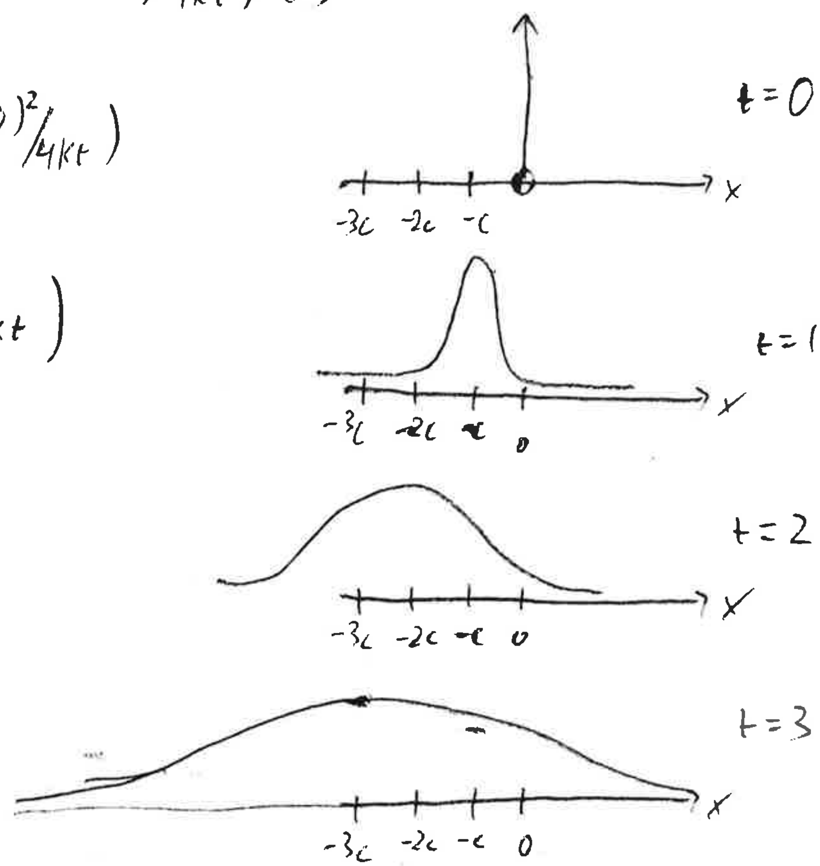
(b) $u(x,0)=f(x)=\delta(x)$

$$F(\omega)=\frac{1}{2\pi}$$

$$u(x,t)=\int_{-\infty}^{\infty}\delta(s)\frac{1}{\sqrt{4\pi kt}}\exp\left(-(x+ct-s)^2/4kt\right)ds$$

$$=\frac{1}{\sqrt{4\pi kt}}\exp\left(-(x+ct-0)^2/4kt\right)$$

$$=\frac{1}{\sqrt{4\pi kt}}\exp\left(-(x+ct)^2/4kt\right)$$



The convection term cu_x moves the solution to the left at speed c.

10.4.4 (a) $u_t = k u_{xx} - \gamma u$, $-\infty < x < \infty$

$u(x,0) = f(x)$

$U(\omega,t) = \mathcal{F}[u(x,t)]$

$U_t = k(-i\omega)^2 U - \gamma U$

$= (-\gamma - k\omega^2) U$

$U(\omega,t) = A(\omega)\exp(-k\omega^2 t - \gamma t)$, $A(\omega)$ an unknown "constant"

Initial data: $u(x,0) = f(x) \Rightarrow U(x,0) = \mathcal{F}[f(x)] = F(\omega)$

$U(\omega,0) = A(\omega)\exp(0) = F(\omega) \Rightarrow A(\omega) = F(\omega)$

$U(\omega,t) = F(\omega)\exp(-k\omega^2 t - \gamma t)$

$= F(\omega)\exp(-k\omega^2 t)\underbrace{\exp(-\gamma t)}$

is a constant with respect to $\omega \Rightarrow$ FT does not affect it.

$u(x,t) = \mathcal{F}^{-1}[F(\omega)\exp(-k\omega^2 t)\exp(-\gamma t)]$

$= \exp(-\gamma t)\,\mathcal{F}^{-1}[F(\omega)\exp(-k\omega^2 t)]$

$= \exp(-\gamma t)\cdot(f * h)(x)$, where $h(x) = \mathcal{F}^{-1}[\exp(-k\omega^2 t)]$

$= \sqrt{\frac{\pi}{kt}}\exp(-x^2/4kt)$

$$= \exp(-\gamma t)\cdot \underbrace{\int_{-\infty}^{\infty} f(s)\frac{1}{\sqrt{4\pi kt}}\exp\left(-(x-s)^2/4kt\right)ds}_{=f*h}$$

(b) The solution suggests that γ only plays a role in the $\exp(-\gamma t)$ factor, multiplying the solution to the standard heat equation. Then suppose we define $v(x,t) = \exp(\gamma t)\,u(x,t)$. What PDE does $v(x,t)$ satisfy?

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial t}\left[\exp(\gamma t)\cdot u(x,t)\right] = \gamma\exp(\gamma t)\cdot u + \exp(\gamma t)\cdot u_t$$

$$= \gamma\exp(\gamma t)u + \exp(\gamma t)\left[k u_{xx} - \gamma u\right]$$

$$= k\exp(\gamma t)\,u_{xx}$$

$$= k\frac{\partial^2}{\partial x^2}\left[\exp(\gamma t)\,u\right]$$

$$= k\,v_{xx}$$

And $v(x,0) = \exp(\gamma\cdot 0)\,u(x,0) = f(x)$.

So $v(x,t)$ solves

$$v_t = k v_{xx}$$

$$v(x,0) = f(x),$$

which is a simpler equation. We could have first solved this and then computed u as $u(x,t) = \exp(-\gamma t)\,v(x,t)$.

10.4.6 $\quad \dfrac{d^2y}{dx^2} - xy = 0, \quad y(x) = Ai(x)$

$$\lim_{|x|\to\infty} y = 0, \quad y(0) = 3^{-2/3}/\Gamma(2/3) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{\omega^3}{3}\right)d\omega$$

Take Fourier transform of ODE: $(-i\omega)^2 Y(\omega) - (-i\,Y'(\omega)) = 0$

$$Y'(\omega) + i\omega^2 Y(\omega) = 0$$

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This is a first-order linear ODE $\Longrightarrow$ it can be solved via integrating factors.

$Y'(\omega) + P(\omega)Y(\omega) = 0 \Longrightarrow P(\omega) = i\omega^2$

integrating factor: $\mu(\omega) = \exp\left(\int P(\omega)\,d\omega\right)$

$= \exp\left(\frac{i}{3}\omega^3\right)$

(1) multiply ODE by $\mu(\omega)$

(2) manipulate left-hand side into a total derivative.

$Y'(\omega) + i\omega^2 Y(\omega) = 0$

$Y'(\omega)\exp\left(\frac{i}{3}\omega^3\right) + Y(\omega)\cdot i\omega^2 \exp\left(\frac{i}{3}\omega^3\right) = 0$

$\frac{d}{d\omega}\left[Y(\omega)\exp\left(\frac{i}{3}\omega^3\right)\right] = 0$

$Y(\omega)\exp\left(\frac{i}{3}\omega^3\right) = c$ (undetermined constant)

$Y(\omega) = c \cdot \exp\left(-\frac{i}{3}\omega^3\right)$

How to compute c?

$y(0) = \frac{1}{\pi}\int_0^\infty \cos(\omega^3/3)\,d\omega$

By definition of FT: $y(0) = \int_{-\infty}^{\infty} Y(\omega)\exp(-i\omega\cdot 0)\,d\omega$

$= \int_{-\infty}^{\infty} Y(\omega)\,d\omega$

$= c \cdot \int_{-\infty}^{\infty} \exp\left(\frac{-i}{3}\omega^3\right) d\omega$

$$\frac{1}{\pi}\int_0^\infty \cos(w^3/3)\,dw = y(0) = c\int_{-\infty}^{\infty} \exp\left(-\frac{i}{3}w^3\right) dw$$

$$= c\cdot\left[\int_{-\infty}^{\infty} \cos(w^3/3)\,dw + i\int_{-\infty}^{\infty} \sin(w^3/3)\,dw\right]$$

$$= c\left[2\cdot\int_0^\infty \cos(w^3/3)\,dw + i\cdot 0\right]$$

because $\cos(x)$ is an even function

because $\sin(x)$ is an odd function

So $\frac{1}{\pi}\int_0^\infty \cos\left(\frac{w^3}{3}\right)dw = 2c\int_0^\infty \cos\left(\frac{w^3}{3}\right)dw$

$$\Longrightarrow c = \frac{1}{2\pi}$$

$$\mathcal{F}[Ai(x)] = Y(w) = \frac{1}{2\pi}\exp\left(-iw^3/3\right)$$

10.4.7 (a) $u_t = k u_{xxx}, \quad -\infty < x < \infty$

$u(x,0) = f(x)$

$U(w,t) = \mathcal{F}[u(x,t)] \Longrightarrow U_t = k(-iw)^3 U$

$= ikw^3 U$

$$U(w,t) = U(w,0)\cdot\exp(ikw^3 t)$$

Initial data $\Longrightarrow U(w,0) = \mathcal{F}[f(x)] = F(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{iwx}\,dx$

$$U(w,t) = F(w)\exp(ikw^3 t)$$

$$u(x,t) = \mathcal{F}^{-1}[U(w,t)] = \mathcal{F}^{-1}[F(w)\exp(ikw^3 t)]$$

$$= \int_{-\infty}^{\infty} e^{ikw^3 t} e^{-iwx} \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s) e^{iws}\,ds\,dw$$

(b) Convolution theorem:

$$U = F \cdot \exp(ikt\omega^3)$$

$$\Longrightarrow u(x,t) = (f * g)(\cdot, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\, g(x-s)\, ds$$

where $g(x,t) = \mathcal{F}^{-1}[\exp(ikt\omega^3)]$

$$= \int_{-\infty}^{\infty} \exp(ikt\omega^3 - i\omega x)\, d\omega$$

(c) Exercise 16.4.6 $\Longrightarrow$ $\mathcal{F}[Ai(x)] = \frac{1}{2\pi}\exp(-i\omega^3/3)$

so $\mathcal{F}^{-1}[\exp(-\frac{i}{3}\omega^3)] = 2\pi Ai(x)$

Define $Y(\omega) = \frac{1}{2\pi}\exp(-i\omega^3/3)$ (FT of Airy function)

Define $G(\omega) = \exp(ikt\omega^3)$ (FT of g)

Goal: relate G to Y, in order to compute $g(x,t)$.

Assume $k > 0$:

$$2\pi Y(-(3kt)^{1/3}\omega) = \exp(-\tfrac{i}{3}(-(3kt)^{1/3}\omega)^3)$$

$$= \exp(-\tfrac{i}{3}\cdot(-3kt)\omega^3) = G(\omega)$$

So $\mathcal{F}^{-1}[G(\omega)] = 2\pi\, \mathcal{F}^{-1}[Y(-(3kt)^{1/3}\omega)]$

$$= 2\pi \int_{-\infty}^{\infty} Y(-(3kt)^{1/3}\omega)\cdot \exp(-i\omega x)\, d\omega$$

$$u = -(3kt)^{1/3}\omega$$

$$du = -(3kt)^{1/3}\, d\omega$$

$$= \frac{-2\pi}{(3kt)^{1/3}}\int_{\infty}^{-\infty} Y(u)\exp\left(-iu\left(\frac{-x}{(3kt)^{1/3}}\right)\right)du$$

$$= \frac{2\pi}{(3kt)^{1/3}}\int_{-\infty}^{\infty} Y(u)\exp\left(-iu\left(\frac{-x}{(3kt)^{1/3}}\right)\right)du$$

$$\mathcal{F}^{-1}[G(\omega)] = \frac{2\pi}{(3kt)^{1/3}} \int_{-\infty}^{\infty} Y(u) \exp\left(-iu\left(\frac{-x}{(3kt)^{1/3}}\right)\right) du$$

$$= \frac{2\pi}{(3kt)^{1/3}}\, y\left(\frac{-x}{(3kt)^{1/3}}\right)$$

$$= \frac{2\pi}{(3kt)^{1/3}}\, Ai\left[\frac{-x}{(3kt)^{1/3}}\right] = g(x,t)$$

So $u(x,t) = f * g$

$$= \int_{-\infty}^{\infty} f(s) \frac{1}{(3kt)^{1/3}}\, Ai\left[\frac{-(x-s)}{(3kt)^{1/3}}\right] ds$$