MATH 3150

Homework 1 solutions.

12.1 (a) $\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x}$

rate of change of energy density

$\frac{\partial \phi}{\partial x}$ ~ net heat flux out of infinitesimal element

negative sign: $\frac{\partial e}{\partial t}$ should be heat flux into element

(b) $\phi = -K_0 \frac{\partial u}{\partial x}$

negative sign: heat flows to left if temperature gradient is positive.

(c) $\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x}$

negative sign: $\frac{\partial \phi}{\partial x}$ is net flux of chemical pollutant out of element change in chemical pollutant is negative of this.

(d) $\phi = -k \frac{\partial u}{\partial x}$

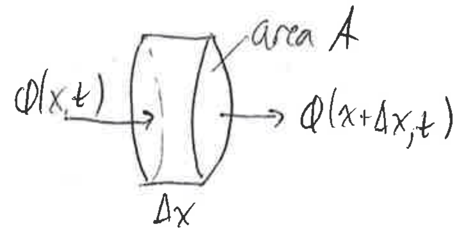
negative sign: pollutant flows in opposite direction of concentration gradient.

12.2 Constant thermal properties: c, ρ, K_0

No sources: $Q = 0$.

(a) Total heat energy: $e(x,t) A \Delta x$

Heat flux: $-A\phi(x+\Delta x, t) + A\phi(x,t)$



Conservation of energy: $\frac{\partial}{\partial t}[e(x,t) A \Delta x] = -A\phi(x+\Delta x,t) + A\phi(x,t)$

$$\frac{\partial e}{\partial t} = -\frac{\phi(x+\Delta x,t) - \phi(x,t)}{\Delta x}$$

as $\Delta x \downarrow 0$: $\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x}$

Relationship between e and u: $e = c\rho u$

Fourier's Law: $\phi = -K_0 \frac{\partial u}{\partial x}$

So $\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} \longrightarrow \frac{\partial u}{\partial t} = \frac{K_0}{c\rho}\frac{\partial^2 u}{\partial x^2}$.

(b)

Conservation of energy: $\frac{d}{dt}\int_a^b e(x,t)\,dx = -\varphi(b,t) + \varphi(a,t)$

$$\int_a^b \frac{\partial e}{\partial t}\,dx = -\int_a^b \frac{\partial \varphi}{\partial x}\,dx \longrightarrow \text{true for all } a, b,$$

therefore: $\frac{\partial e}{\partial t} = -\frac{\partial \varphi}{\partial x}$

Remainder same as part (a)

12.4 Essentially the same as 12.2 (see book for meaning of φ, u).

12.6 (a) Heat energy per unit mass required to change temperature by Δu degrees is $c(x,u)\Delta u$ (assume $c(x,u)$ is constant for $u \to u+\Delta u$).

Summing over all u from 0 to $u \longrightarrow \int_0^u c(x,s)\,ds$.

(b) (12.3) still holds: $\frac{\partial e}{\partial t} = -\frac{\partial \varphi}{\partial x} + Q$ since definition of c does not affect that relation.

Fourier's law is unchanged: $\varphi = -K_0 \frac{\partial u}{\partial x}$.

But now: $e(x,t) = \rho(x)\int_0^u c(x,s)\,ds$.

And $\frac{\partial}{\partial t} e = \frac{\partial}{\partial t}\left[\rho(x)\int_0^{u(x,t)} c(x,s)\,ds\right] = \rho(x)\frac{\partial}{\partial t}\int_0^{u(x,t)} c(x,s)\,ds = \rho(x)\,c(x,u)\,\frac{\partial u}{\partial t}$

Fundamental Thm of calc.

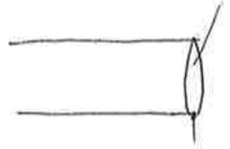
Heat equation: $\rho(x)\,c(x,u)\frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2}$

12.8 $c\rho u$: energy per unit volume.

Total energy: $\int_0^L c\rho u A\,dx$.

#1.3.1

temperature $u(L,t)$



temperature $u_B(t)$ (known)

$x=L$

heat flux proportional to temperature difference

$$\downarrow$$

$$\phi(L,t) = H[u(L,t) - u_B(t)]$$

if $u(L,t) - u_B(t) > 0$, then heat should flow to right, so $H > 0$.

$$\phi(L,t) = -K_0(L)\frac{\partial u}{\partial x}(L,t) = H[u(L,t) - u_B(t)].$$

#1.3.2 Heat energy flowing out of x_{0-} equals flow into x_{0+}

$$\downarrow$$

$$\phi(x_0-) = \phi(x_0+)$$

$$-K_0(x_0-)\frac{\partial u}{\partial x}(x_0-) = -K_0(x_0+)\frac{\partial u}{\partial x}(x_0+)$$

$\frac{\partial u}{\partial x}$ is continuous if $\frac{\partial u}{\partial x}(x_0+) = \frac{\partial u}{\partial x}(x_0-)$, which holds if $K_0(x_0-) = K_0(x_0+)$, i.e. if thermal diffusivity K_0 is continuous at x_0.