Eigenvalue methods for DE's

MATH 2250 Lecture 35
Book section 7.3

November 26, 2019

Linear, constant-coefficient DE systems

In this section we will consider the following linear, constant-coefficient, homogeneous DE system:

$$\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x},$$

where $\boldsymbol{A}$ is a *constant* matrix.
Our goal in this section is to compute the general solution to this equation.

Solutions via an ansatz

$$\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x},$$

We take inspiration from our scalar DE experience: we make the ansatz

$$\boldsymbol{x}(t) = \boldsymbol{v}e^{\lambda t},$$

for an unknown constant λ and vector $\boldsymbol{v}$.

What condition must $(\lambda, \boldsymbol{v})$ satisfy so that this is a solution?

Solutions via an ansatz

$$\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x},$$

We take inspiration from our scalar DE experience: we make the ansatz

$$\boldsymbol{x}(t) = \boldsymbol{v}e^{\lambda t},$$

for an unknown constant λ and vector $\boldsymbol{v}$.
What condition must $(\lambda, \boldsymbol{v})$ satisfy so that this is a solution?

$$\lambda \boldsymbol{v} e^{\lambda t} = \boldsymbol{A}\boldsymbol{v}e^{\lambda t}$$
$$\lambda \boldsymbol{v} = \boldsymbol{A}\boldsymbol{v}.$$

Thus, our ansatz is a solution if and only if $(\lambda, \boldsymbol{v})$ is an eigenpair of $\boldsymbol{A}$.

Eigenvalue method

$$\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x},$$

Thus, we can compute the general solution to this DE via the following algorithm:

- Compute the eigenvalues of $\boldsymbol{A}$.
- Compute n linearly independent eigenvectors of $\boldsymbol{A}$, if possible.
- If the above is possible, then $\boldsymbol{x}_j(t) = \boldsymbol{v}_j e^{\lambda_j t}$ for $j = 1, \ldots, n$ are n linearly independent solutions to the DE, and hence superposition yields the general solution.

Example

Example (Example 7.3.1)

Compute the general solution to the DE system

$$\begin{aligned} x_1'(t) &= 4x_1 + 2x_2 \\ x_2'(t) &= 3x_1 - x_2 \end{aligned}$$

Example

Example (Example 7.3.1)

Compute the general solution to the DE system

$$\begin{aligned} x_1'(t) &= 4x_1 + 2x_2 \\ x_2'(t) &= 3x_1 - x_2 \end{aligned}$$

Example

Compute the general solution to the DE system

$$\begin{aligned} x_1'(t) &= 3x_1 - 2x_2 \\ x_2'(t) &= -x_1 + 3x_2 - 2x_3 \\ x_3'(t) &= -x_2 + 3x_3 \end{aligned}$$

Complex eigenvalues

Complex eigenvalues can yield complex-valued solutions.

To obtain real-valued solutions, we combine real and imaginary parts (just as we did for scalar DE's).

Example (Example 7.3.3)

Solve the initial value problem,

$$\begin{aligned} x_1'(t) &= 4x_1 - 3x_2 \\ x_2'(t) &= 3x_1 + 4x_2 \end{aligned}$$

with initial data $x_1(0) = 1$ $x_2(0) = -1$.