Properties of linear systems of DE's

MATH 2250 Lecture 34
Book section 7.2

November 25, 2019

Vector- and matrix-valued functions (1/2)

Before continuing with DE's, we first note that vectors and matrices can be functions.

For example, we can have

$$\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \qquad \boldsymbol{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}$$

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These functions have **continuity** or **differentiability**, if each of their components is continuous or differentiable, respectively.

Vector- and matrix-valued functions (2/2)

If $\boldsymbol{A}(t)$, and $\boldsymbol{B}(t)$ are matrix-valued functions, $\boldsymbol{C}$ is a constant matrix, and c is a scalar, then the following differentiation properties hold:

- $\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{A}+\boldsymbol{B}) = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{A} + \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{B}$
- $\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{A}\boldsymbol{B}) = \boldsymbol{A}\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}\boldsymbol{B}$
- $\frac{\mathrm{d}}{\mathrm{d}t}(c\boldsymbol{A}) = c\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}$
- $\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{C}\boldsymbol{A}) = \boldsymbol{C}\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}$
- $\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{A}\boldsymbol{C}) = \frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}\boldsymbol{C}$

First-order systems of linear DE's

We consider the following first-order linear system:

$$\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x},$$

where $\boldsymbol{x}(t)$ is a size-n vector, and $\boldsymbol{A}$ is an $n \times n$ matrix.

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The general first-order, linear, nonhomogeneous DE that we consider is

$$\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{f}(t).$$

If $\boldsymbol{f} \equiv 0$, then the equation is homogeneous.

Superposition

Most of our scalar DE theory extends to systems. We start with the **principle of superposition**.

If $\boldsymbol{x}_1(t), \ldots, \boldsymbol{x}_n(t)$ are all solutions to the *homogeneous* DE

$$\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x},$$

then

$$\boldsymbol{x}(t) = \sum_{j=1}^{n} c_j \boldsymbol{x}_j(t) = c_1 \boldsymbol{x}_1(t) + c_2 \boldsymbol{x}_2(t) + \cdots + c_n \boldsymbol{x}_n(t),$$

is also a solution to the DE.

Example

Example (Example 7.2.2)

Verify that

$$\boldsymbol{x}_1(t) = \begin{pmatrix} 3e^{2t} \\ 2e^{2t} \end{pmatrix}, \qquad \boldsymbol{x}_2(t) = \begin{pmatrix} e^{-5t} \\ 3e^{-5t} \end{pmatrix}$$

solve the DE

$$\boldsymbol{x}'(t) = \begin{pmatrix} 4 & -3 \\ 6 & -7 \end{pmatrix}.$$

Also, compute a solution $\boldsymbol{x}(t)$ satisfying $\boldsymbol{x}(0) = \begin{pmatrix} 2 & 1 \end{pmatrix}^T$.

Linear independence (1/2)

Linear independence for vector-valued functions is defined as expected:
The size-n vector-valued functions $\boldsymbol{x}_1(t), \ldots, \boldsymbol{x}_n(t)$ are **linearly independent** on the interval I if the equation

$$c_1\boldsymbol{x}_1(t) + \cdots + c_n\boldsymbol{x}_n(t) = \mathbf{0},$$

is true for all t in I *only* when $c_1 = \cdots = c_n = 0$.
Otherwise, they are linearly dependent.

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Linear independence can be explicitly tested with the **Wronskian**. The scalar function

$$W(t) := \det \left(\begin{array}{cccc} | & | & \cdots & | \\ \boldsymbol{x}_1(t) & \boldsymbol{x}_2(t) & \cdots & \boldsymbol{x}_n(t) \\ | & | & \cdots & | \end{array} \right)$$

is the Wronskian of $\boldsymbol{x}_1(t), \ldots, \boldsymbol{x}_n(t)$.

Linear independence (2/2)

If $\boldsymbol{x}_1(t), \ldots, \boldsymbol{x}_n(t)$ are solutions to a linear, homogeneous system of n DE's, then

- $W(t) \equiv 0$ for all t in I if and only if the functions are linearly dependent on I.
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Example (Example 7.2.3)

Verify that the three functions

$$\boldsymbol{x}_1(t) = \begin{pmatrix} 2e^t \\ 2e^t \\ e^t \end{pmatrix}, \qquad \boldsymbol{x}_2(t) = \begin{pmatrix} 2e^{3t} \\ 0 \\ -e^{3t} \end{pmatrix}, \qquad \boldsymbol{x}_3(t) = \begin{pmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{pmatrix},$$

are linearly independent on $I = \mathbb{R}$.

Solutions to homogeneous DE's

From linearly independent solutions, we obtain general solutions:

Theorem

Consider the size-n system of DE's $\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}$, and suppose that $\boldsymbol{A}$ is continuous for every t in I. If $\boldsymbol{x}_1(t), \ldots, \boldsymbol{x}_n(t)$ are a linearly independent set of solutions, then

$$\boldsymbol{x}(t) = \sum_{j=1}^{n} c_j \boldsymbol{x}_j(t)$$

is the ***general solution*** *to the DE for arbitrary $c_1, \ldots, c_n$. (I.e., there are no other solutions.)*

Solutions to non-homogeneous DE's

General solutions to non-homogeneous solutions work similarly:

Theorem

Consider the size-n system of non-homogeneous DE's $\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{f}(t)$, and suppose that $\boldsymbol{A}$ is continuous for every t in I. Suppose that $\boldsymbol{x}_1(t), \ldots, \boldsymbol{x}_n(t)$ are a linearly independent set of solutions to the homogeneous DE $\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t)$, and suppose that $\boldsymbol{x}_p(t)$ is any solution to the non-homogeneous DE. Tnen

$$\boldsymbol{x}(t) = \boldsymbol{x}_p(t) + \sum_{j=1}^{n} c_j \boldsymbol{x}_j(t)$$

*is the **general solution** to the non-homogeneous DE for arbitrary $c_1, \ldots, c_n$. (I.e., there are no other solutions.)*

Example

Example (Example 7.2.5)

Consider the DE system initial value problem

$$\begin{array}{rclllll} x'(t) & = & 3x & -2y & & -9t & 13, \\ y'(t) & = & -x & 3y & -2z & 7t & -15, \\ z'(t) & = & & -y & 3z & -6t & 7, \end{array} \quad , \quad \begin{array}{rcl} x_1(0) & = & 0 \\ x_2(0) & = & 1 \\ x_3(0) & = & -3 \end{array}$$

With $\boldsymbol{x} = (x, y, z)^T$, the functions

$$\boldsymbol{x}_1(t) = \begin{pmatrix} 2e^t \\ 2e^t \\ e^t \end{pmatrix}, \qquad \boldsymbol{x}_2(t) = \begin{pmatrix} 2e^{3t} \\ 0 \\ -e^{3t} \end{pmatrix}, \qquad \boldsymbol{x}_3(t) = \begin{pmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{pmatrix},$$

are solutions to *homogeneous* DE, and

$$\boldsymbol{x}_p(t) = \begin{pmatrix} 3t \\ 5 \\ 2t \end{pmatrix},$$

is a solution to the non-homogeneous DE. Solve the initial value problem.