Eigenvalues and eigenvectors

MATH 2250 Lecture 31
Book section 6.1

November 15, 2019

Solution methods for DE's

Given a second-order, linear, constant-coefficient DE,

$$ay''(x) + by'(x) + cy(x) = f(x), \qquad y(0) = y_0, \; y'(0) = y_0',$$

we have two ways to solve this equation:

- "Directly": the characteristic equation, undetermined coefficients
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The next couple of weeks:

There is a third way we investigate solving such equations: through linear algebra.

As usual, we need some background before discussing DE's.

Eigenvectors

Recall: we use uppercase boldface letters ($\boldsymbol{A}$) to denote matrices, and lowercase boldface letters ($\boldsymbol{v}$) to denote vectors.

Let $\boldsymbol{A}$ be an $n \times n$ (square!) matrix. Then a <u>non-zero</u> size-n vector $\boldsymbol{v}$ is an **eigenvector** of $\boldsymbol{A}$ if

$$\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v},$$

for some *scalar* value λ. (Any scalar value is fine, even 0.)

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Example (Example 6.1.1)

Show that $\boldsymbol{v} = (2,1)^T$ is an eigenvector of the matrix

$$\boldsymbol{A} = \begin{pmatrix} 5 & -6 \\ 2 & -2 \end{pmatrix}$$

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Note: the condition $\boldsymbol{v} \neq \mathbf{0}$ is important. Eigenvectors *cannot* be the zero vector $\mathbf{0}$. (Because $\mathbf{0}$ always satisfies $\boldsymbol{A}\mathbf{0} = \mathbf{0}$.)

Eigenvalues

Suppose $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{A}$. Then we have

$$\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}.$$

The scalar λ that makes this equation true is called an **eigenvalue** of $\boldsymbol{A}$. (In the previous example, $\lambda = 2$ is an eigenvalue of $\boldsymbol{A}$.)

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Eigenvalues can be 0. (This is informative.)
Eigenvectors cannot be $\mathbf{0}$. ($\boldsymbol{v} = \mathbf{0}$ is not informative.)

Eigenvectors are not unique: if $\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}$, then $2\boldsymbol{v}$ also satisfies this equation.

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Example

Let V be the set of eigenvectors of the matrix $\boldsymbol{A}$ with eigenvalue λ. Show that V is a subspace.
(This subspace V is called an **eigenspace** of $\boldsymbol{A}$.)

The characteristic equation (1/2)

How do we compute eigenvalues and eigenvectors?
Given $\boldsymbol{A}$, we must find $(\lambda, \boldsymbol{v})$ satisfying

$$\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v} = \lambda \boldsymbol{I} \boldsymbol{v},$$

where $\boldsymbol{I}$ is the identity matrix.

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Rewriting this, we have

$$(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{v} = \boldsymbol{0}.$$

Recall $\boldsymbol{v}$ is not zero: thus, $\boldsymbol{A} - \lambda\boldsymbol{I}$ must be a matrix which multiplies a <u>non-zero</u> vector to result in $\boldsymbol{0}$.

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Thus, $\boldsymbol{A} - \lambda \boldsymbol{I}$ must be a *singular* matrix.

The characteristic equation (2/2)

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Recall that this is equivalent to:

$$\det\left(\boldsymbol{A} - \lambda \boldsymbol{I}\right) = 0$$

This turns out to be a polynomial equation in λ, whose roots define eigenvalues.
The equation $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$ is the **characteristic equation** of $\boldsymbol{A}$.

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How do we find eigenvectors? If λ is a root of the characteristic equation, then there must be a non-zero $\boldsymbol{v}$ such that

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Example (Example 6.1.2)

Compute all the eigenvalues and eigenvectors for

$$\boldsymbol{A} = \begin{pmatrix} 5 & 7 \\ -2 & -4 \end{pmatrix}$$

More examples (1/2)

Example

Compute the eigenvalues and eigenvectors of

$$\boldsymbol{A} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

(Eigenvalues/eigenvectors can be complex!)

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Example (Example 6.1.4)

Compute the eigenvalues and eigenvectors of

$$\boldsymbol{A} = \boldsymbol{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(Root multiplicity of the characteristic equation can correspond to dimensions of eigenspaces.)

More examples (2/2)

Example (Example 6.1.6)

Compute the eigenvalues and eigenvectors of

$$\boldsymbol{A} = \begin{pmatrix} 3 & 0 & 0 \\ -4 & 6 & 2 \\ 16 & -15 & -5 \end{pmatrix}$$

(Computations are essentially the same for $n \times n$ matrices.)

More examples (2/2)

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Example

Compute the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

(Root multiplicity of characteristic equation need not correspond to dimensions of eigenspaces.)